Theorem 4.5.15. *Let $m \geq 1$ be an integer. Let $R > 0$ and suppose that f is analytic on the punctured disk $0 < |z - z_0| < R$. Then the following conditions are equivalent:*
(i) f has a pole of order m at z_0.
(ii) There is an $r > 0$ and there is a nowhere vanishing analytic function ϕ on $B_r(z_0)$ such that

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}, \qquad \text{when } 0 < |z - z_0| < R.$$

(iii) There exists a complex number $\alpha \neq 0$ such that

$$\lim_{z \to z_0} (z - z_0)^m f(z) = \alpha.$$

(iv) There exist $r > 0$ and complex numbers a_n for $n \geq -m$ such that $a_{-m} \neq 0$ and

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

for all $z \in B_r(z_0) \setminus \{z_0\}$.

Proof. $(i) \Rightarrow (ii)$. If f has a pole of order m at z_0, then the function g in (4.5.3) has a zero of order m at z_0. By Theorem 4.5.2, there is a nowhere vanishing function λ on a neighborhood $B_r(z_0)$ of z_0 such that $g(z) = (z - z_0)^m \lambda(z)$. This implies assertion (ii) with $\phi = 1/\lambda$.

$(ii) \Rightarrow (iii)$. This is an easy consequence of the observation that $(z - z_0)^m f(z) = \phi(z)$ which converges to $\phi(z_0)$ as $z \to z_0$. Then $\alpha = \phi(z_0) \neq 0$ works.

$(iii) \Rightarrow (iv)$. Using Theorem 4.5.12(iii) we deduce that z_0 is a removable singularity of $(z - z_0)^m f(z)$ on the annulus $|z - z_0| < r$. Expanding $q(z) = (z - z_0)^m f(z)$ in a power series centered at z_0, we write

$$q(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \quad |z - z_0| < r, \tag{4.5.4}$$

for some c_k with $c_0 = \alpha$. Dividing both sides of (4.5.4) by $(z - z_0)^m$ and using that $(z - z_0)^m f(z) = q(z)$ for $0 < |z - z_0| < r$, we obtain the claimed Laurent series of f. Note that $a_k = c_{k+m}$ for $k \geq -m$ and that $a_{-m} = \alpha \neq 0$.

$(iv) \Rightarrow (i)$. Factoring $\frac{1}{(z-z_0)^m}$ from the Laurent series of $f(z)$ in (iv) we write $f(z)$ as

$$\frac{1}{(z - z_0)^m} \left[a_{-m} + a_{-m+1}(z - z_0) + \cdots + a_{-1}(z - z_0)^{m-1} + a_0(z - z_0)^m + \cdots \right]$$

and we note that the expression inside the square brackets is an analytic function on $|z - z_0| < r$. This function, which we call h, satisfies $h(z_0) = a_{-m} \neq 0$, thus there is a neighborhood $B_\delta(z_0)$ of z_0 on which it is nowhere vanishing. We have

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \frac{1}{|z - z_0|^m} \lim_{z \to z_0} |h(z)| = \infty \cdot |a_{-m}| = \infty$$