Proof. The closure $\overline{\Omega}$ of a region Ω is the region Ω together with its boundary. If Ω is bounded, then $\overline{\Omega}$ is closed and bounded. Suppose that f is a function as in the statement of the theorem and suppose that it has infinitely many zeros in Ω; then by the Bolzano-Weierstrass theorem, there is an infinite sequence of zeros, $\{z_n\}_{n=1}^{\infty}$ in Ω that converges to a point z_0 in $\overline{\Omega}$. Since f is continuous in $\overline{\Omega}$, $f(z_0) = \lim_{n\to\infty} f(z_n) = 0$, and since f is nonvanishing on the boundary, we conclude that z_0 lies in Ω. Hence by Theorem 4.5.5, f is identically zero on Ω, and since f is continuous, f must be zero on the boundary, which is a contradiction. Hence f can have at most finitely many zeros in Ω. ■

Isolated Singularities

If a function is analytic in a neighborhood of a point z_0 except possibly at the point z_0, then z_0 is called an **isolated singularity** of the function. There are only three different types of isolated singularities, which are defined as follows.

Definition 4.5.8. Suppose that z_0 is an isolated singularity of an analytic function f defined in a deleted neighborhood of z_0. Then
(i) z_0 is a **removable singularity** of f if the function can be redefined at z_0 to be analytic there.
(ii) z_0 is a **pole** of f if $\lim\limits_{z\to z_0} |f(z)| = \infty$.
(iii) z_0 is an **essential singularity** of f if it is neither a pole nor a removable singularity.

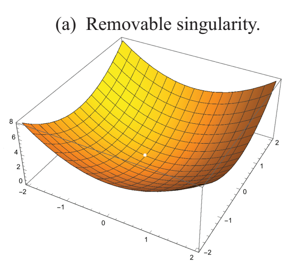

Fig. 4.16 Near a removable singularity, $|f|$ is bounded.

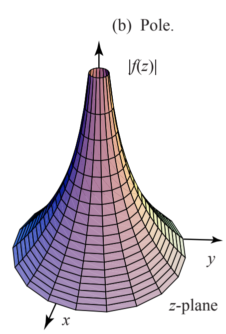

Fig. 4.17 Near a pole, $|f|$ tends to infinity.

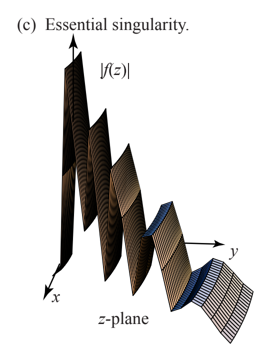

Fig. 4.18 Near an essential singularity, $|f|$ is neither bounded nor tends to ∞. Its graph behaves erratically.

When redefining f at a removable singularity z_0, we set $f(z_0) = \lim_{z\to z_0} f(z)$; otherwise, f would not be continuous and hence could not be analytic at z_0. Note that when z_0 is a removable singularity, f must be bounded near z_0 (Figure 4.16).

The singularity is a pole if the graph of $|f|$ blows up to infinity as we approach z_0 (Figure 4.17). It is harder to explain the graph of an essential singularity, but as the