Example 4.5.3. (Order of zeros) Find the order m of the zero of $\sin z$ at $z_0 = 0$; then express $\sin z = z^m \lambda(z)$, where λ is analytic and satisfies $\lambda(0) \neq 0$.

Solution. Clearly, 0 is a zero of $\sin z$. The order of the zero is equal to the order of the first nonvanishing derivative of $f(z) = \sin z$ at 0. Since $f'(z) = \cos z$ and $\cos 0 = 1 \neq 0$, we conclude that the order of the zero at 0 is 1. We have for all z

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right) = z\lambda(z),$$

where $\lambda(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$. The function $\lambda(z)$ is entire (because it is a convergent power series for all z), and $\lambda(0) = 1$. Also, for $z \neq 0$, $\lambda(z) = \frac{\sin z}{z}$, which is entire by Example 4.3.10. $\square$

We now address the following question: Suppose that a function is analytic and not identically zero on a region. Can it have zeros that are not isolated? The answer is no, and this depends on the fact that the region is connected.

Theorem 4.5.4. *All zeros of a nonconstant analytic function on a region are isolated.*

Proof. Let f be an analytic function defined on a region Ω. Define the subsets

$$\Omega_0 = \{z \in \Omega : \ f(z) = 0 \text{ and } z \text{ is not isolated}\}$$
$$\Omega_1 = \{z \in \Omega : \ f(z) \neq 0\} \cup \{z \in \Omega : \ f(z) = 0 \text{ and } z \text{ is isolated}\}.$$

Then Ω_0 and Ω_1 are disjoint sets whose union is Ω. We show that Ω_0 and Ω_1 are open; then by the connectedness of Ω (Proposition 2.1.7) we have either $\Omega = \Omega_0$ or $\Omega = \Omega_1$. If $\Omega = \Omega_0$, then every point in Ω is a zero of f; in this case f vanishes everywhere on Ω. If $\Omega = \Omega_1$ then every point of Ω is either not a zero or an isolated zero of f. In this case all zeros of f are isolated.

If $w \in \Omega_0$, then assertion (ii) in Theorem 4.5.2 does not hold, hence assertion (i) must hold and thus f vanishes identically in some neighborhood $B_r(w)$. Clearly, all the points in $B_r(w)$ are not isolated zeros of f, so $B_r(w)$ is contained in Ω_0. This shows that Ω_0 is open in this case. Now let $w \in \Omega_1$. If $f(w) \neq 0$, then there is a neighborhood of w on which f is nonvanishing; this neighborhood is contained in Ω_1. If $f(w) = 0$ and w is an isolated zero, then Theorem 4.5.2 guarantees the existence of a neighborhood $B_\delta(w)$ of w on which f has no zeros except the isolated one at w. This neighborhood is also contained in Ω_1, since $B'_\delta(w) = B_\delta(w) \setminus \{w\}$ is contained in $\{z \in \Omega : \ f(z) \neq 0\}$. Thus Ω_1 is open. $\blacksquare$

Theorem 4.5.5. (Identity Principle) *Suppose that f, g are analytic functions on a region Ω. If $\{z_n\}_{n=1}^\infty$ is an sequence of distinct points in Ω with $f(z_n) = g(z_n)$ for all n and $z_n \to z_0 \in \Omega$, then $f(z) = g(z)$ for all $z \in \Omega$.*

Proof. If z_n is a zero of $f - g$ in Ω and $z_n \to z_0$ as $n \to \infty$, then $f(z_n) = g(z_n)$ for all n and by continuity it follows that