

The ideas of the previous example are the basis for the techniques of Chapter 5: To compute integrals around circles, we find Laurent **series** expansions and integrate term by term. Only the term involving $\frac{1}{z}$ survives.

Exercises 4.4

In Exercises 1–18, find the Laurent series of the function in the indicated annulus.

1. $\frac{1}{1+z}$, $1 < |z|$
2. $\frac{1}{2+iz}$, $2 < |z|$
3. $\frac{1}{3+2iz}$, $\frac{5}{2} < |z+i|$
4. $\frac{1}{1+i-z}$, $\sqrt{2} < |z-2|$
5. $\frac{1}{1+z^2}$, $1 < |z|$
6. $\frac{1}{1-z^2}$, $1 < |z-2| < 3$
7. $\frac{3+z}{2-z}$, $2 < |z|$
8. $\frac{1+z}{1-z}$, $1 < |z|$
9. $z + \frac{1}{z}$, $1 < |z-1|$
10. $z^{22}e^{\frac{1}{z^2}}$, $0 < |z|$
11. $\coth z$, $0 < |z| < \pi$
12. $\cot z$, $0 < |z| < \pi$
13. $\frac{z}{(z+2)(z+3)}$, $2 < |z| < 3$
14. $\frac{-2}{(2z-1)(2z+1)}$, $\frac{1}{2} < |z|$
15. $\frac{1}{(3z-1)(2z+1)}$, $\frac{1}{3} < |z| < \frac{1}{2}$
16. $\frac{1}{2z^2-3z+1}$, $1 < |z|$
17. $\frac{z^2+(1-i)z+2}{(z-i)(z+2)}$, $1 < |z| < 2$
18. $\frac{4z-5}{(z-2)(z-1)}$, $1 < |z-2|$

In Exercises 19–22, find all Laurent series expansions of the function at the indicated point z_0 .

19. $\frac{1}{z+i}$, $z_0 = 1$
20. $\frac{1}{z^2-1}$, $z_0 = 2$
21. $\frac{1}{(z-1)(z+i)}$, $z_0 = -1$
22. $\frac{1}{z^2+1}$, $z_0 = 1+i$

23. (a) Derive the Laurent series

$$\frac{1}{1+z} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{z^n} \quad 1 < |z|.$$

Starting with this Laurent series, find the Laurent series of the following functions in the annulus $1 < |z|$:

$$(b) \frac{1}{(1+z)^2}; \quad (c) \frac{z}{(1+z)^2}; \quad (d) \frac{z^2}{(1+z)^3}.$$

24. Find the Laurent series of $\csc^2 z$ in the annulus $0 < |z| < \pi$.

In Exercises 25–30, evaluate the integral using an appropriate Laurent series. As usual, we denote by $C_R(z_0)$ the positively oriented circle of radius $R > 0$ and center z_0 .

25. $\int_{C_1(0)} \sin \frac{1}{z} dz$
26. $\int_{C_1(0)} \frac{\cos \frac{1}{z}}{z} dz$
27. $\int_{C_1(0)} \cos z \sin \frac{1}{z} dz$
28. $\int_{C_1(0)} e^{z^2+\frac{1}{z}} dz$
29. $\int_{C_4(0)} \operatorname{Log} \left(1 + \frac{1}{z}\right) dz$
30. $\int_{C_1(0)} z^{10} e^{\frac{1}{z}} dz$

31. Suppose that f is analytic on a region Ω . Let $\overline{B_R(z_0)}$ be a closed disk contained in Ω .

- (a) For $n = 0, 1, 2, \dots$, derive the Laurent series expansion