which converge uniformly on all closed subannuli of $A_{R_1,R_2}(z_0)$. Multiply both sides of the preceding identity by $(z-z_0)^{-k-1}$ for some arbitrary integer k and integrate over the circle $C_R(z_0)$. We obtain

$$\frac{1}{2\pi i}\int_{C_R(z_0)}\sum_{n\in\mathbb{Z}} a_n(z-z_0)^{n-k-1}dz = \frac{1}{2\pi i}\int_{C_R(z_0)}\sum_{n\in\mathbb{Z}} b_n(z-z_0)^{n-k-1}dz.$$

But both series above converge uniformly on all closed annuli containing the circle $C_R(z_0)$ and contained in the annulus $A_{R_1,R_2}(z_0)$, so the integration and summation can be interchanged ~~via~~ by Corollary 4.1.6. We obtain

$$\sum_{n\in\mathbb{Z}}\frac{1}{2\pi i}\int_{C_R(z_0)} a_n(z-z_0)^{n-k-1}dz = \sum_{n\in\mathbb{Z}}\frac{1}{2\pi i}\int_{C_R(z_0)} b_n(z-z_0)^{n-k-1}dz$$

and noting that all terms of the series vanish except for the terms with $n=k$, in view of the identity

$$\frac{1}{2\pi i}\int_{C_R(z_0)}(z-z_0)^{-1}dz = 1$$

we deduce that $a_k = b_k$ for all k. ■

As with power series, often in computing Laurent series, we can avoid using (4.4.2) by resorting to known series.

Example 4.4.2. (Laurent series centered at 0) The function $e^{\frac{1}{z}}$ is analytic in the (degenerated) annulus $0<|z|$, with center at $z_0=0$. Find its Laurent series expansions in this annulus.

Solution. Start with the exponential series $e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$, which is valid for all z. In particular, if $z\neq 0$, putting $\frac{1}{z}$ into this series, we obtain

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{1}{n!z^n} = 1+\sum_{n=1}^{\infty}\frac{1}{n!z^n} = 1+\frac{1}{1!z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\cdots.$$

By the uniqueness of the Laurent series representation in the annulus $0<|z|$, we have thus found the Laurent series of $e^{\frac{1}{z}}$ in the annulus $0<|z|$. Note that the series has infinitely many negative powers of z. □

Example 4.4.3. (Laurent series centered at 0) The function $\frac{1}{1-z}$ is analytic in the annulus $1<|z|$, with center at $z_0=0$. Find its Laurent series expansions in this annulus.

Solution. Here we use the geometric series $\sum_{n=0}^{\infty}w^n=\frac{1}{1-w}$ for $|w|<1$. Factor z from the denominator and use the fact that $1<|z|$, then

$$\frac{1}{1-z} = \frac{1}{z}\frac{1}{\frac{1}{z}-1} = \frac{-1}{z}\frac{1}{1-\frac{1}{z}} = \frac{-1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{-1}{z^{n+1}}, \qquad 1<|z|.$$