$$
\begin{aligned}
c_n &= \sum_{k=0}^{n-1} \frac{B_k}{k!}\,\frac{1}{(n-k)!} \qquad \text{(because } a_0 = 0\text{)}\\
&= \frac{1}{n!}\sum_{k=0}^{n-1} \frac{n!}{k!(n-k)!}B_k = \frac{1}{n!}\sum_{k=0}^{n-1}\binom{n}{k}B_k.
\end{aligned}
$$

By the uniqueness of the power series expansion, (4.3.9) implies that $c_1 = 1$, and $c_n = 0$ for all $n \geq 2$. Thus,

$$
\begin{aligned}
&c_1 = 1 \quad \Rightarrow \quad \frac{1}{1!}B_0 = 1 \quad \Rightarrow \quad B_0 = 1;\\
&c_n = 0, \quad n \geq 2 \quad \Rightarrow \quad \frac{1}{n!}\sum_{k=0}^{n-1}\binom{n}{k}B_k = 0, \quad n \geq 2.
\end{aligned}
$$

Changing n to $n+1$ in the last identity, we see that, for $n \geq 1$,

$$
\frac{1}{(n+1)!}\sum_{k=0}^{n}\binom{n+1}{k}B_k = 0,
$$

which implies that

$$
\frac{1}{(n+1)!}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k + \frac{1}{(n+1)!}\binom{n+1}{n}B_n = 0.
$$

Now, realizing that $\binom{n+1}{n} = n+1$, we deduce (4.3.8).

(d) With the aid of a computer and the recurrence relation (4.3.8) we generated the Bernoulli numbers shown in Table 2.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$-\frac{691}{2730}$

Table 2. Bernoulli numbers.

(e) As the table suggests, $B_{2n+1} = 0$ for $n \geq 1$. This is clearly a fact about an even function. Consider $f(z) - B_1 z$, i.e., eliminate the linear term of the Maclaurin series:

$$
\frac{z}{e^z - 1} + \frac{z}{2} = \frac{z + ze^z}{2(e^z - 1)} = \frac{z(e^{\frac{z}{2}} + e^{-\frac{z}{2}})}{2(e^{\frac{z}{2}} - e^{-\frac{z}{2}})} = \frac{z}{2}\coth\left(\frac{z}{2}\right). \tag{4.3.10}
$$

This is an even function. Hence all the odd numbered coefficients in its Maclaurin series are zero (Exercise 29). This implies that $B_{2n+1} = 0$ for all $n \geq 1$. □

Using (4.3.10) and the Maclaurin series of $\frac{z}{e^z-1}$, we see that for $|z| < 2\pi$