Example 4.3.12. (Bernoulli numbers) Let $f(z) = \frac{z}{e^z-1}$, $z \neq 0$, $f(0) = 1$.
(a) Show that f is analytic at 0.
(b) By Theorem 4.3.1, f has a Maclaurin series expansion. Show that its radius of convergence is $R = 2\pi$.
(c) Write the Maclaurin series in the form

$$f(z) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \qquad |z| < 2\pi.$$

The number B_n is called the nth **Bernoulli number**. Show that $B_0 = 1$, and derive the recurrence relation

$$B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k, \quad n \geq 1. \tag{4.3.8}$$

(d) Find $B_0, B_1, B_2, \ldots, B_{12}$, with the help of the recursion formula and a calculator.
(e) Show that $B_{2n+1} = 0$ for $n \geq 1$. Here $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient.

Solution. (a) Consider $g(z) = \frac{1}{f(z)} = \frac{e^z-1}{z}$ for $z \neq 0$, and $g(0) = 1$. By Theorem 4.3.11, g is analytic at 0. Since $g(0) \neq 0$, $\frac{1}{g} = f$ is therefore analytic at $z = 0$.
(b) The Maclaurin series of f converges in the largest disk around $z_0 = 0$ on which f is defined and is analytic. Away from zero, f is analytic on the set where $e^z - 1 \neq 0$. Since $e^z = 1$ precisely when z is an integer multiple of $2\pi i$, we see that the Maclaurin series converges for all $|z| < 2\pi$, and the radius of convergence is 2π.

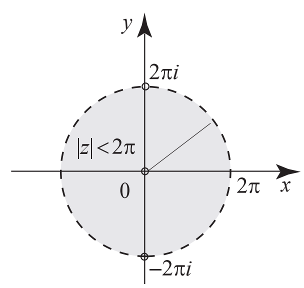

Fig. 4.8 The disk of convergence of $\frac{z}{e^z-1}$.

(c) Multiplying both sides of the Maclaurin series expansion of $\frac{z}{e^z-1}$ by $e^z - 1$ and using the Maclaurin series $e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=1}^{\infty} \frac{z^n}{n!}$, we obtain

$$z = (e^z - 1) \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \sum_{n=1}^{\infty} \overbrace{\frac{z^n}{n!}}^{a_n z^n} \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \sum_{n=1}^{\infty} c_n z^n, \qquad |z| < 2\pi, \tag{4.3.9}$$

where c_n will be computed from the Cauchy product formula (Theorem 1.5.28). Note that because we are multiplying by the power series of $e^z - 1$ whose first term is z, the first term in the Cauchy product will have degree greater than or equal to 1 (thus $c_0 = 0$). We have for each $n \geq 1$