$$\left|(z-z_0)^n \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\right| \le M \frac{\rho^n}{r^{n+1}} = \frac{M}{r}\left(\frac{\rho}{r}\right)^n = M_n.$$

Since $\rho/r < 1$, the series $\sum M_n$ converges and it follows from the Weierstrass M-test that the series

$$\frac{f(\zeta)}{\zeta - z} = \sum_{n=0}^{\infty} (z-z_0)^n \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}$$

converges uniformly for all ζ on $C_r(z_0)$. Integrating term by term both sides of the equality

$$\frac{1}{2\pi i}\frac{f(\zeta)}{\zeta - z} = \frac{1}{2\pi i}\sum_{n=0}^{\infty} (z-z_0)^n \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}$$

and using Cauchy's generalized integral formula, we deduce

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= \sum_{n=0}^{\infty} (z-z_0)^n \overbrace{\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta}^{\frac{f^{(n)}(z_0)}{n!}} \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n, \end{aligned}$$

which completes the first part of the proof. The uniqueness of the power series is a consequence of Theorem 4.2.10. ■

Remark 4.3.3. Since there is no guarantee that Ω is the largest region on which the function f in Theorem 4.3.1 is defined, the series (4.3.2) may converge on an open disk $B_{R_1}(z_0)$ larger than $B_R(z_0)$. Since a Taylor series is a power series, it follows from Theorem 4.2.5(iii) that the series (4.3.2) converges to $f(z)$ absolutely on $B_{R_1}(z_0)$ and uniformly on all closed subdisks of $B_R(z_0)$ (in fact all closed subdisks of $B_{R_1}(z_0)$). Also, in view of Corollary 4.2.9, the series may be differentiated term-by-term in $B_R(z_0)$ as many times as we wish. This yields the Taylor series representation of the nth derivative

$$f^{(n)}(z) = \sum_{j=n}^{\infty} \frac{f^{(j)}(z_0)}{(j-n)!}(z-z_0)^{j-n}, \qquad |z-z_0| < R. \tag{4.3.4}$$

Example 4.3.4. (Maclaurin series of e^z, $\cos z$, and $\sin z$) Find the Maclaurin series expansions of (a) e^z, (b) $\cos z$, (c) $\sin z$.

Solution. We first note that all three functions are entire, so the Maclaurin series will converge for all z; that is, $R = \infty$ in all three cases.