$$c_k = \frac{f^{(k)}(z_0)}{k!} \qquad (k = 0, 1, 2, \ldots), \tag{4.2.7}$$

with the usual convention: $f^{(0)}(z) = f(z)$ and $0! = 1$. This formula relates the coefficients of the power series to the derivatives of f at the center of the series and is known as ~~the~~ **Taylor formula**. A consequence of this formula is the following useful result.

Theorem 4.2.10. (Uniqueness of Power Series Expansions) *Let $R > 0$. If for all z satisfying $|z - z_0| < R$ we have*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n ,$$

then $a_n = b_n$ for all n. In particular, if $\sum_{n=0}^{\infty} c_n (z - z_0)^n = 0$ for all $|z - z_0| < R$, then $c_n = 0$ for all $n = 0, 1, 2, \ldots$.

Proof. By Taylor's formula, we have $a_n = \frac{f^{(n)}(z_0)}{n!} = b_n$. The second assertion of the theorem follows by seting $b_n = 0$. ■

Power series can be integrated term-by-term over paths contained in their disk of convergence.

Example 4.2.11. (Term-by-term integration) Start with the geometric series

$$\frac{1}{1+\zeta} = \sum_{n=0}^{\infty} (-1)^n \zeta^n \qquad |\zeta| < 1.$$

Since the integral of an analytic function on a disk is independent of path, integrating both sides from 0 to z, where $|z| < 1$, we obtain

$$\int_{[0,z]} \frac{1}{1+\zeta}\, d\zeta = \sum_{n=0}^{\infty} (-1)^n \int_{[0,z]} \zeta^n \, d\zeta .$$

Using the independence of paths (Theorem 3.3.4) and the fact that $\frac{d}{dz} \operatorname{Log}(1+z) = \frac{1}{1+z}$ and $\operatorname{Log} 1 = 0$, we obtain

$$\operatorname{Log}(1+z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} \qquad |z| < 1. \qquad \square$$

Example 4.2.12. (Term-by-term differentiation) Find the sum $\sum_{n=1}^{\infty} n z^n$ and determine its radius of convergence.

Solution. The series looks like the derivative of the geometric series