Thus

$$\max|f_n(z)g(z)-f(z)g(z)|\le M\max|f_n(z)-f(z)|\to 0,$$

because f_n converges uniformly to f on E. Thus $f_ng\to fg$ uniformly on E. To prove (ii), apply (i) to the sequence of partial sums $u_1+\cdots+u_n$ which converges uniformly to u. ■

Proof (Theorem 4.1.10). (i) The function f is continuous by Theorem 4.1.3(i). To prove that f is analytic, we apply Morera's theorem (Theorem 3.8.10). Let γ be an arbitrary closed ~~triangular~~ path lying in a closed disk in Ω. It is enough to show that $\int_\gamma f(z)\,dz=0$. We have $\int_\gamma f_n(z)\,dz=0$ for all n, by Cauchy's theorem (Theorem 3.5.4), because f_n is analytic inside and on γ; and by Theorem 4.1.5, $\int_\gamma f_n(z)\,dz\to\int_\gamma f(z)\,dz$ as $n\to\infty$. So $\int_\gamma f(z)\,dz=0$ and (i) follows.
(ii) Let $z_0\in\Omega$ and let $\overline{B_R(z_0)}$ be a closed disk contained in Ω, centered at z_0 with radius $R>0$, with positively oriented boundary $C_R(z_0)$. Since $f_n\to f$ uniformly on $C_R(z_0)$ and $\frac{1}{(z-z_0)^{k+1}}$ is continuous on $C_R(z_0)$, it follows from Lemma 4.1.11(i) that

$$\frac{f_n(z)}{(z-z_0)^{k+1}}\to\frac{f(z)}{(z-z_0)^{k+1}}$$

uniformly for all z on $C_R(z_0)$. Applying Theorem 4.1.5 and using the generalized Cauchy integral formula, we deduce

$$f_n^{(k)}(z_0)=\frac{k!}{2\pi i}\int_{C_R(z_0)}\frac{f_n(z)}{(z-z_0)^{k+1}}\,dz\to\frac{k!}{2\pi i}\int_{C_R(z_0)}\frac{f(z)}{(z-z_0)^{k+1}}\,dz=f^{(k)}(z_0),$$

which proves (ii). ■

Theorem 4.1.10 may fail if we replace analytic functions by differentiable functions of a real variable. That is, if E is a subset of the real line and $f_n(x)\to f(x)$ uniformly on E, it does not follow in general that f_n' converges to f', as the next example shows.

Example 4.1.12. (Failure of termwise differentiation) For $0\le x\le 2\pi$ and $n=1,2,\dots$ define $f_n(x)=\frac{e^{inx}}{in}$. It is clear that $f_n\to 0$ uniformly on $[0,2\pi]$. But $f_n'(x)=e^{inx}$ and this sequence does not converge except at $x=0$ or $x=2\pi$. (See Example 1.5.9) Consequently, f_n' does not converge to 0. Can we understand how this occurs within the larger framework of complex functions? Replace x by z and consider the sequence functions $f_n(z)=\frac{e^{inz}}{in}$. We cannot find a complex neighborhood of the real interval $[0,2\pi]$ where f_n converges, as such a neighborhood would contain z with $\operatorname{Im} z<0$. Thus Theorem 4.1.10 does not apply. □

Corollary 4.1.13. *Suppose that $\{u_n\}_{n=1}^\infty$ is a sequence of analytic functions on a region Ω and that $u=\sum_{n=1}^\infty u_n$ converges uniformly on every closed disk in Ω. Then u is analytic on Ω. Moreover, for all integers $k\ge 1$, the series may be differentiated term by term k times to yield*