is near c and so it is bounded. For $|z| \leq M$, $|f|$ is bounded because it is a continuous function on a closed and bounded set.]

15. Suppose that f is entire and $\lim_{z\to\infty} f(z) = 0$. Show that f is identically 0.

16. (a) Suppose that f is entire and f' is bounded in $\mathbb{C}$. Show that $f(z) = az + b$ for all $z \in \mathbb{C}$.
(b) Show that if f is entire and $f^{(n)}$ is bounded, then f is a polynomial of degree at most n.

17. Suppose that f is entire and omits ~~an~~ a nonempty open set, i.e., there is an open disk $B_R(w_0)$ with $R > 0$ in the w-plane such that $f(z)$ does not lie in $B_R(w_0)$ for all z. Show that f is constant. [Hint: Consider $g(z) = \frac{1}{f(z)-w_0}$ and show that you can apply Liouville's theorem.] (A deep result in complex analysis known as Picard's theorem asserts that an entire nonconstant function can omit at most one value.)
18. Suppose that f is entire. Show that if either $\operatorname{Re} f$ or $\operatorname{Im} f$ are bounded, then f is constant. [Hint: Use Exercise 17.]

19. Suppose that f is entire and $\lim_{z\to\infty} f(z)/z = 0$. Show that f is constant. [Hint: Use Cauchy's estimate to show that $f'(z) = 0$.]

20. Suppose that f is entire and $\lim_{z\to\infty} \frac{f(z)}{z} = c$, where c is a constant. Show that $f(z) = cz + b$. [Hint: Apply the result of the previous exercise $g(z) = f(z) - cz$.]

21. A function $f(z) = f(x+iy)$ is called **doubly periodic** if there are real numbers $T_1 > 0$ and $T_2 > 0$ such that $f(x+T_1+iy) = f(x+i(y+T_2)) = f(x+iy)$ for all $z = x+iy$ in $\mathbb{C}$. Show that if a function is entire and doubly periodic then it is constant. Can an entire function f be periodic in one of x or y without being constant?

22. What conclusion do you draw from Corollary 3.9.3 about the function e^{z^2}?

23. (a) Suppose that f is analytic in a bounded region Ω and continuous on the boundary of Ω. Suppose that $|f|$ is constant on the boundary of Ω. Show that either f has a zero in Ω or f is constant in Ω.
(b) Find all analytic functions f on the unit disk such that $|z| < |f(z)|$ for all $|z| < 1$ and $|f(z)| = 1$ for all $|z| = 1$. Justify your answer.

24. Let f and g be analytic functions on the open unit disk $B_1(0)$ and continuous and nonvanishing on the closed disk $\overline{B_1(0)}$. Suppose that $|f(z)| = |g(z)|$ for all $|z| = 1$. Show that $f(z) = A\,g(z)$ for all $|z| \leq 1$, where A is a constant such that $|A| = 1$.

25. Suppose that f is analytic on $|z| < 1$ and continuous on $|z| \leq 1$. Suppose that $f(z)$ is real-valued for all $|z| = 1$. Show that f is constant for all $|z| \leq 1$. [Hint: Consider $g(z) = e^{if(z)}$.]

26. Suppose that f and g are analytic in a bounded region Ω and continuous on the boundary of Ω. Suppose that g does not vanish in $\overline{\Omega}$ and $|f(z)| \leq |g(z)|$ for all z on the boundary of Ω. Show that $|f(z)| \leq |g(z)|$ for all z in Ω.