Clearly, $\Omega = \Omega_0 \cup \Omega_1$, and Ω_0 and Ω_1 are disjoint, and Ω_1 is nonempty because $|f|$ is assumed to attain its maximum in Ω. The set Ω_0 is open because $|f|$ is continuous (Exercise 41, Section 2.2). If we show that Ω_1 is also open, then, as Ω is open and connected, it cannot be written as the union of two disjoint open nonempty sets (Proposition 2.1.7). This will force Ω_0 to be empty. Consequently, $\Omega = \Omega_1$, implying that $|f| = M$ is constant in Ω.

So let us prove that Ω_1 is open. Pick z in Ω_1. Since Ω is open, we can find an open disk $B_\delta(z)$ in Ω, centered at z with radius $\delta > 0$. We will show that $B_\delta(z)$ is contained in Ω_1. This will imply that Ω_1 is open. Let $0 < r < \delta$ as shown in Figure 3.80. Using (3.9.5) and the fact that $|f(z)| = M$, we obtain

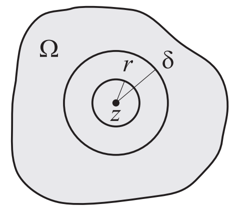

Fig. 3.80 We have that $B_r(z) \subset B_\delta(z) \subset \Omega$.

$$M = |f(z)| \le \frac{1}{2\pi}\int_0^{2\pi} \overbrace{\left|f(z+re^{it})\right|}^{\le M} dt \le \frac{1}{2\pi}\int_0^{2\pi} M\,dt = M.$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(z+re^{it})\right| dt = M,$$

and Lemma 3.9.5(ii) implies that

$$\left|f(z+re^{it})\right| = M$$

for all t in $[0, 2\pi]$. This shows that $C_r(z)$, the circle of radius r and center at z, is contained in Ω_1. But this is true for all r satisfying $0 < r < \delta$, and this implies that $B_\delta(z)$ is contained in Ω_1. ■

Suppose that f is analytic in Ω and continuous on the boundary of Ω. By Theorem 3.9.6, $|f|$ cannot attain its maximum inside Ω unless f is constant. This leads us to the following two questions.

- Does $|f|$ attain its maximum on the boundary of Ω?
- If $|f(z)| \le M$ on the boundary of Ω, can we infer that $|f(z)| \le M$ for all z in Ω?

The next example shows that in general the answers to both questions are negative.

Example 3.9.7. (The maximum modulus principle on an unbounded region)

Let $\Omega = \{z : \operatorname{Re} z > 0,\ \operatorname{Im} z > 0\}$ be the first quadrant, bounded by the semi-infinite nonnegative x and y-axes.