The following lemma about real-valued functions states an obvious fact: If the values of a function are less than or equal to some constant M and if the average of the function is equal to M, then the function must be identically equal to M.

Lemma 3.9.5. *(i) Suppose that h is a continuous real-valued function such that $h(t) \geq 0$ for all t in $[a,b]$ $(a<b)$. If*

$$\int_a^b h(t)\,dt = 0,$$

then $h(t)=0$ for all t in $[a,b]$.
(ii) Suppose that h is a continuous real-valued function such that $h(t) \leq M$ (alternatively, $h(t) \geq M$) for all t in $[a,b]$. If

$$\frac{1}{b-a}\int_a^b h(t)\,dt = M,$$

then $h(t)=M$ for all t in $[a,b]$.

Proof. (i) The proof is by contradiction. Assume that $h(t_0)=\delta>0$ for some t_0 in (a,b). Since h is continuous, we can find an interval (c,d) that contains t_0 with $(c,d)\subset(a,b)$ and $h(t)>\delta/2$ for all t in (c,d). Since the integral of a nonnegative function increases if we increase the interval of integration, we conclude that

$$\int_a^b h(t)\,dt \geq \int_c^d h(t)\,dt \geq \frac{\delta}{2}(d-c) > 0,$$

which is a contradiction. Hence $h(t)=0$ for all t in (a,b) and, by continuity of h, h is also 0 at the endpoints a and b.
(ii) We consider the case $h \leq M$ only as the other case is similar. Let $k=M-h$. Then k is a continuous and nonnegative function. Note that

$$\int_a^b k(t)\,dt = 0 \Leftrightarrow \frac{1}{b-a}\int_a^b (M-h(t))\,dt = 0 \Leftrightarrow \frac{1}{b-a}\int_a^b h(t)\,dt = M.$$

As the last equation holds, k must have vanishing integral, and from part (i) it follows that k is identically equal to zero; thus h is identically equal to M. ■

Theorem 3.9.6. (Maximum Modulus Principle) *Suppose that f is analytic on a region Ω. If $|f|$ attains a maximum in Ω, then f is constant in Ω.*

Proof. The connectedness property of Ω is crucial in the proof. Suppose that $|f|$ attains a maximum in Ω. If we show that $|f|$ is constant, then by Exercise 36, Section 2.5, it will follow that f is constant. Let $M=\max_{z\in\Omega}|f(z)|$,

$$\Omega_0 = \{z\in\Omega:\ |f(z)|<M\}$$
$$\Omega_1 = \{z\in\Omega:\ |f(z)|=M\}.$$