circle $C_R(z_0)$, we fix $0 < r < R$ and work on a disk of radius r on which f is analytic (see Figure 3.79). Applying Theorem 3.8.6, for any $n \in \mathbb{N}$, we write

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta. \quad (3.9.2)$$

For ζ on $C_r(z_0)$, we have

$$|\zeta - z_0| = r$$

and so it follows that

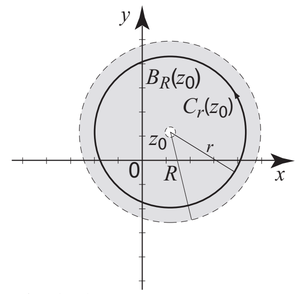

Fig. 3.79 Picture of the proof.

$$\left|\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\right| = \frac{|f(\zeta)|}{|\zeta - z_0|^{n+1}} = \frac{|f(\zeta)|}{r^{n+1}} \le \frac{M}{r^{n+1}}. \quad (3.9.3)$$

Applying the ML-inequality [i.e., estimate (3.2.31)] to the integral on the right side of (3.9.2) and using (3.9.3) we find that

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi}\frac{M}{r^{n+1}}\ell(C_r(z_0)) = M\frac{n!}{r^n}.$$

Since this holds for all $0 < r < R$, letting $r \to R$, we deduce (3.9.1). ■

The following surprising application of Cauchy's estimate was proved by the French mathematician Joseph Liouville (1809–1882). Recall that a function is called entire if it is analytic on all of $\mathbb{C}$.

Theorem 3.9.2. (Liouville's Theorem) *A bounded entire function must be constant.*

Proof. Let f be an entire function satisfying $|f(z)| \le M$ for all $z \in \mathbb{C}$. Applying Cauchy's estimate to f' on a disk of radius $R > 0$ around a point $z_0 \in \mathbb{C}$, we obtain $|f'(z_0)| \le \frac{M}{R}$. Letting $R \to \infty$, we obtain that $f'(z_0) = 0$. Since z_0 is arbitrary, it follows that $f'(z) = 0$ for all z, and hence f is constant as ~~a consequence of~~ by Theorem 2.5.7. ■

Here is one useful application of Liouville's theorem.

Corollary 3.9.3. *If f is entire and $|f(z)| \to \infty$ as $|z| \to \infty$, then f must have at least one zero.*

Proof. Suppose to the contrary that f has no zeros in $\mathbb{C}$. Then $g = 1/f$ is also entire and $|g(z)| \to 0$ as $|z| \to \infty$. This property implies that g is bounded on $\mathbb{C}$ (Exercise 14). By Theorem 3.9.2, g is constant and consequently f is constant. ■

The exponential function e^z is entire and never equals to 0. As a consequence of Corollary 3.9.3, we deduce that $|e^z|$ does not tend to infinity as $|z| \to \infty$, and indeed the $\lim_{z->\infty} e^z$ does not exist, although $e^{|z|}$ tends to infinity as $|z| \to \infty$.