$$\left|\frac{\phi(z,\zeta)-\phi(z_0,\zeta)}{z-z_0}-\frac{d\phi}{dz}(z_0,\zeta)\right|\le 2M\,\frac{|z-z_0|}{R^2}. \tag{3.8.9}$$

Integrating the expression inside the absolute value on the left in (3.8.9) and using (3.8.7) we find

$$\begin{aligned}\left|\frac{g(z)-g(z_0)}{z-z_0}-\int_C\frac{d\phi}{dz}(z_0,\zeta)d\zeta\right| &= \left|\frac{\int_C[\phi(z,\zeta)-\phi(z_0,\zeta)]d\zeta}{z-z_0}-\int_C\frac{d\phi}{dz}(z_0,\,\zeta)d\zeta\right|\\ &= \left|\int_C\left[\frac{\phi(z,\zeta)-\phi(z_0,\zeta)}{z-z_0}-\frac{d}{dz}\phi(z_0,\zeta)\right]d\zeta\right|\\ &\le \ell(C)\,2M\,\frac{|z-z_0|}{R^2},\end{aligned}$$

where we have used the ML-inequality and (3.8.9). As $z\to z_0$, the difference quotient on the left side of the inequality approaches $g'(z_0)$, while the right side of the inequality tends to 0. This proves (3.8.8). ■

Generalized Cauchy Integral Formula

We now use Theorem 3.8.5 to deduce the generalized Cauchy integral formula.

Theorem 3.8.6. (Generalized Cauchy Integral Formula) *Suppose that f is analytic on a region Ω that contains a positively oriented simple closed path C and its interior. Then f has derivatives of any order at all points z in the interior of C given by*

$$f^{(n)}(z)=\frac{n!}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta. \tag{3.8.10}$$

Proof. When $n=0$, $f^{(0)}=f$ and $0!=1$; in this case identity (3.8.10) was proved in Theorem 3.8.1. Let U be the interior of C. Assuming by induction that (3.8.10) holds for a natural number n, for $z\in U$ and $\zeta\in C$ define

$$\phi(z,\zeta)=\frac{n!}{2\pi i}\frac{f(\zeta)}{(\zeta-z)^{n+1}}\,,$$

and note that $\phi(z,\zeta)$ is analytic in z in U and continuous in $\zeta\in C$. Moreover, for $z\in U$ we have

$$\frac{d\phi}{dz}(z,\zeta)=\frac{(n+1)!}{2\pi i}\frac{f(\zeta)}{(\zeta-z)^{n+2}}\,,$$

which is continuous in $\zeta\in C$. Applying Theorem 3.8.5 we obtain that $f^{(n+1)}$ is analytic in U and

$$f^{(n+1)}(z)=\frac{(n+1)!}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)^{n+2}}\,d\zeta$$