Proof. Using (3.8.1), for $0 < |z - z_0| < \frac{R}{2}$, we write

$$f(z) = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \text{and} \quad f(z_0) = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{f(\zeta)}{\zeta - z_0}\, d\zeta.$$

Combining these integrals and simplifying, we obtain

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)}\, d\zeta. \tag{3.8.5}$$

For ζ on $C_R(z_0)$ and $0 < |z - z_0| < \frac{R}{2}$ we have $|\zeta - z| > \frac{R}{2}$, and so $\frac{1}{|\zeta - z|} < \frac{2}{R}$. Thus

$$\left| \frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)} - \frac{f(\zeta)}{(\zeta - z_0)^2} \right| = \left| \frac{f(\zeta)(z - z_0)}{(\zeta - z)(\zeta - z_0)^2} \right| \le M \frac{2}{R} \frac{|z - z_0|}{R^2} = 2M \frac{|z - z_0|}{R^3}.$$

It follows from the ML-inequality that

$$\left| \frac{1}{2\pi i} \int_{C_R(z_0)} \left[\frac{f(\zeta)}{(\zeta - z)(\zeta - z_0)} - \frac{f(\zeta)}{(\zeta - z_0)^2} \right] d\zeta \right| \le \frac{2\pi R}{|2\pi i|} \frac{2M|z - z_0|}{R^3} = 2M \frac{|z - z_0|}{R^2}.$$

Separating the integrals and using identity (3.8.5) we obtain

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^2}\, d\zeta \right| \le 2M \frac{|z - z_0|}{R^2}. \tag{3.8.6}$$

Letting $z \to z_0$ in (3.8.6) we deduce (3.8.4). Combining (3.8.4) and (3.8.6) yields (3.8.3). ∎

Theorem 3.8.5. *Let C be a path and let U be an open set. Let $\phi(z, \zeta)$ be a function defined for $z \in U$ and $\zeta \in C$. Suppose that $\phi(z, \zeta)$ is continuous in $\zeta \in C$ and analytic in $z \in U$ and that the complex derivative $\frac{d\phi}{dz}(z, \zeta)$ is continuous in $\zeta \in C$. Then the function*

$$g(z) = \int_C \phi(z, \zeta)\, d\zeta \tag{3.8.7}$$

is analytic in U and its derivative is

$$g'(z) = \int_C \frac{d\phi}{dz}(z, \zeta)\, d\zeta. \tag{3.8.8}$$

Proof. For a fixed $z_0 \in U$ choose $R > 0$ so that $\overline{B_R(z_0)}$ is contained in U. Let

$$M = \max_{z \in \overline{B_R(z_0)}, \zeta \in C} |\phi(z, \zeta)|.$$

Since continuous functions attain a maximum on compact (i.e., closed and bounded) sets, we have $M < \infty$. For each ζ, by assumption $\phi(z, \zeta)$ is an analytic function of z in U. For z such that $0 < |z - z_0| < \frac{R}{2}$ and $\zeta \in C$, Lemma 3.8.4 implies