$$\int_{C_{1/4}(i)} \frac{f(z)}{z-i}\,dz = -\pi.$$

For the second integral, we have

$$\int_{C_{1/4}(-i)} \frac{e^{\pi z}}{(z-i)(z+i)}\,dz = \int_{C_{1/4}(-i)} \frac{g(z)}{z+i}\,dz = 2\pi i g(-i),$$

where $g(z) = \frac{e^{\pi z}}{z-i}$, and so $g(-i) = \frac{e^{-i\pi}}{-2i} = \frac{1}{2i} = -\frac{i}{2}$. Hence

$$\int_{C_{1/4}(-i)} \frac{g(z)}{z-i}\,dz = \pi.$$

Adding the two integrals together, we find that

$$\int_{C_2(0)} \frac{e^{\pi z}}{(z-i)(z+i)}\,dz = 0. \qquad \square$$

Cauchy's integral formula (3.8.1) shows that the values of $f(z)$, for $z$ inside the path $C$, are determined by the values of $f$ on the curve $C$, and the way to recapture the values inside $C$ is to integrate $f(\zeta)$ against the function $1/[2\pi i(\zeta - z)]$ on $C$. Something analogous is valid for the derivatives of $f$. To achieve this we need to know how to differentiate under the integral sign.

## *Differentiation Under the Integral Sign*

We focus on the analyticity (and continuity) of a function of the form

$$g(z) = \int_C \phi(z,\zeta)\,d\zeta,$$

~~ss~~ where $\zeta$ lies on a ~~simple closed curve~~ path $C$ and $z$ lies in some open set. For instance in Theorem 3.8.1 we had $\phi(z,\zeta) = \frac{1}{2\pi i}\frac{f(\zeta)}{\zeta - z}$. We begin with a lemma.

**Lemma 3.8.4.** *Suppose that $f$ is analytic on an open set containing the closed disk $\overline{B_R(z_0)}$ and satisfies $|f(z)| \le M$ for all $z \in \overline{B_R(z_0)}$. Then for $0 < |z - z_0| < \frac{R}{2}$ the following are valid*

$$\left| \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \right| \le 2M\,\frac{|z-z_0|}{R^2} \tag{3.8.3}$$

*and*

$$f'(z_0) = \frac{1}{2\pi i}\int_{C_R(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^2}\,d\zeta\,. \tag{3.8.4}$$