(b) In evaluating $\int_{C_2(1)} \frac{z^2+3z-1}{(z+3)(z-2)}\,dz$, we first note that the integrand is not analytic at the points $z=-3$ and $z=2$. Only the point $z=2$ is inside the curve $C_2(1)$. So if we let $f(z)=\frac{z^2+3z-1}{z+3}$ the integral takes the form

$$\int_{C_2(1)} \frac{f(z)}{z-2}\,dz = 2\pi i f(2) = \frac{18\pi}{5} i,$$

by the Cauchy integral formula, applied at $z_0 = 2$. □

Some integrals require multiple applications of Cauchy's formula along with applications of Cauchy's theorem. We illustrate this situation with an example.

Example 3.8.3. Compute

$$\int_{C_2(0)} \frac{e^{\pi z}}{z^2+1}\,dz.$$

Solution. Since

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)},$$

the integral cannot be computed directly from Cauchy's formula, since the path contains both $\pm i$ in its interior. To overcome this difficulty, draw small nonintersecting circles inside $C_2(0)$ around $\pm i$, say $C_{1/4}(i)$ and $C_{1/4}(-i)$, as illustrated in Figure 3.8.3. Since $\frac{e^{\pi z}}{z^2+1}$ is analytic in a region containing the interior of $C_2(0)$ and the exterior of the smaller circles, by Cauchy's theorem for multiple connected domains (Theorem 3.7.2), we have

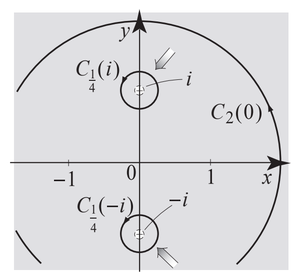

Fig. 3.71 ~~Fig 2.~~ The integral over the outer path C is equal to the sum of the integrals over the inner non-overlapping circles.

$$\int_{C_2(0)} \frac{e^{\pi z}}{z^2+1}\,dz = \int_{C_{1/4}(i)} \frac{e^{\pi z}}{z^2+1}\,dz + \int_{C_{1/4}(-i)} \frac{e^{\pi z}}{z^2+1}\,dz.$$

Now, the two integrals on the right can be evaluated with the help of Cauchy's integral formula (3.8.2). For the first one, we apply Cauchy's formula (3.8.2) with $f(z)=\frac{e^{\pi z}}{z+i}$ and $z_0 = i$, and obtain

$$\int_{C_{1/4}(i)} \frac{e^{\pi z}}{(z-i)(z+i)}\,dz = \int_{C_{1/4}(i)} \frac{f(z)}{z-i}\,dz = 2\pi i f(i).$$

Since $f(i) = \frac{e^{i\pi}}{2i} = \frac{-1}{2i} = \frac{i}{2}$, we get