For an arbitrary complex number $w$ we have $w+\overline{w}=2\,\mathrm{Re}\,w$ and thus we obtain

$$z_1\,\overline{z_2}+\overline{z_1}\,z_2=z_1\,\overline{z_2}+\overline{z_1\,\overline{z_2}}=2\,\mathrm{Re}\,(z_1\,\overline{z_2}).$$

Using this interesting identity, along with (1.2.8) and basic properties of complex conjugation, we obtain

$$\begin{aligned}
|z_1+z_2|^2 &= (z_1+z_2)\overline{(z_1+z_2)}=(z_1+z_2)(\overline{z_1}+\overline{z_2})\\
&= z_1\,\overline{z_1}+z_2\,\overline{z_2}+z_1\,\overline{z_2}+\overline{z_1}\,z_2=|z_1|^2+|z_2|^2+z_1\,\overline{z_2}+\overline{z_1}\,z_2\\
&= |z_1|^2+|z_2|^2+2\,\mathrm{Re}\,(z_1\,\overline{z_2})\\
&\le |z_1|^2+|z_2|^2+2|z_1\,\overline{z_2}| \qquad \text{(by (1.2.14))}\\
&= |z_1|^2+|z_2|^2+2|z_1|\,|\overline{z_2}|\\
&= |z_1|^2+|z_2|^2+2|z_1|\,|z_2| \quad \text{(by (1.2.9) and } |\overline{z_2}|=|z_2|)\\
&= (|z_1|+|z_2|)^2,
\end{aligned}$$

and (1.2.16) follows upon taking square roots on both sides. Next, notice that (1.2.17) is obtained by a repeated applications of (1.2.16), while (1.2.18) is deduced from (1.2.16) replacing $z_2$ by $-z_2$.

Replacing $z_1$ by $z_1-z_2$ in (1.2.16), we obtain $|z_1|\le|z_1-z_2|+|z_2|$, and so

$$|z_1-z_2|\ge|z_1|-|z_2|.$$

Reversing the roles of $z_1$ and $z_2$, and realizing that $|z_2-z_1|=|z_1-z_2|$, we also have

$$|z_1-z_2|\ge|z_2|-|z_1|.$$

Putting these two together, we conclude $|z_1-z_2|\ge\big||z_1|-|z_2|\big|$, which proves inequality in (1.2.20). Finally, we deduce (1.2.19) replacing $z_2$ by $-z_2$. ■

The triangle inequality is used extensively in proofs to provide estimates on the sizes of complex-valued expressions. We illustrate such applications via examples.

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**Example 1.2.9. (Estimating the size of an absolute value)** What is an upper bound for $|z^5-4|$ if $|z|\le 1$?

**Solution.** Applying the triangle inequality, we get

$$|z^5-4|\le|z^5|+4=|z|^5+4\le 1+4=5,$$

because $|z|\le 1$. Hence if $|z|\le 1$, an upper bound for $|z^5-4|$ is 5.

Can we find a number smaller than 5 that is also an upper bound, or is 5 the *least upper bound*? It is easy to see that for $z=-1$, we have $|z^5-4|=|(-1)^5-4|=|-1-4|=5$. Thus, the upper bound 5 is best possible. You should be cautioned that, in general, the triangle inequality is considered a crude inequality, which means that it will not yield least upper bound estimates as it did in this case. See Exercise 37 for an illustration of this fact. □