where the second integral follows from Example 3.2.10. The same argument applies for (3.7.2) but in this case the outcome is zero in view of Example 3.2.10. □

In what follows, we illustrate the applications of the formula in Example 3.7.4.

Example 3.7.5. (Integral of a rational function) Let C be a positively oriented, simple, closed path containing i in its interior and $-i$ in its exterior; see Figure 3.63. Evaluate

$$\int_C \frac{dz}{z^2+1}.$$

Solution. The first thing to check is whether $f(z) = \frac{1}{z^2+1}$ is analytic inside and on C. If it is, then we can apply Cauchy's theorem and be done. Clearly f is not analytic at the points $z = \pm i$, where the denominator z^2+1 vanishes. Since one of these values is inside C, we apply Theorem 3.7.2, as in the previous example or, better yet, use the result of the previous example. The method uses partial fractions and goes as follows. Since $z^2+1 = (z-i)(z+i)$, we have the partial fraction decomposition

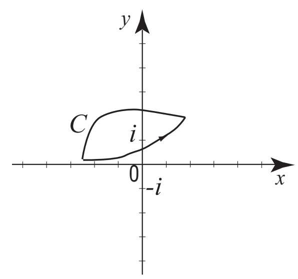

Fig. 3.63 The curve C

$$\frac{1}{z^2+1} = \frac{A}{z-i} + \frac{B}{z+i}. \tag{3.7.3}$$

Thus

$$\int_C \frac{1}{z^2+1}\,dz = A\overbrace{\int_C \frac{dz}{z-i}}^{2\pi i} + B\overbrace{\int_C \frac{dz}{z+i}}^{0} = 2\pi i A,$$

where both integrals after the first equality follow from Example 3.7.4. To complete the evaluation of the integral, we must determine A. To this end, multiply both sides of (3.7.3) by $(z-i)$; then evaluate the equation at $z = i$:

$$\frac{1}{z+i}\Big|_{z=i} = \Big[A + \frac{B}{z+i}(z-i)\Big]\Big|_{z=i};$$

$$\frac{1}{2i} = A + 0;\ A = -\frac{i}{2}.$$

Using this value of A, we get

$$\int_C \frac{dz}{z^2+1} = 2\pi i\Big(-\frac{i}{2}\Big) = \pi.$$ □