neighborhood of ~~either~~ both Γ_1 and Γ_2. Theorem 3.6.7 is now applicable and yields that
$$\int_{\Gamma_1} f(z)\,dz = 0 \qquad \text{and} \qquad \int_{\Gamma_2} f(z)\,dz = 0.$$
Adding these two equalities, we obtain
$$\int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = 0,$$
or
$$\int_P f(z)\,dz + \int_Q f(z)\,dz + \sum_{j=1}^{n}\Big(\int_{P_j^*} f(z)\,dz + \int_{Q_j^*} f(z)\,dz\Big) = 0,$$
using Proposition 3.2.12(iii). Proposition 3.2.12(ii) now yields
$$\int_{P_j^*} f(z)\,dz + \int_{Q_j^*} f(z)\,dz = -\int_{P_j} f(z)\,dz - \int_{Q_j} f(z)\,dz = -\int_{C_j} f(z)\,dz,$$
hence we conclude that
$$\int_P f(z)\,dz + \int_Q f(z)\,dz - \sum_{j=1}^{n}\int_{C_j} f(z)\,dz = 0,$$
which is equivalent to (3.7.1). ■

Next, in the fundamental integral of Example 3.2.10, we replace the circle by an arbitrary positively oriented, simple, closed path C that does not contain a fixed point z_0.

Example 3.7.4. Let C be a positively oriented, simple, closed path, and z_0 be a point not on C. Then
$$\int_C \frac{1}{z-z_0}\,dz = \begin{cases} 0 & \text{if } z_0 \text{ lies in the exterior of } C; \\ 2\pi i & \text{if } z_0 \text{ lies in the interior of } C. \end{cases}$$
Moreover, for $n \neq 1$,
$$\int_C \frac{1}{(z-z_0)^n}\,dz = 0. \tag{3.7.2}$$

Solution. If z_0 lies in the exterior of C, then $\frac{1}{z-z_0}$ is analytic inside and on C and hence the integral is 0, by Theorem 3.6.7 (Instead of Theorem 3.6.7 we may also apply Theorem 3.4.4 since $\frac{1}{z-z_0}$ has continuous derivatives.)

To deal with the case where z_0 lies in the interior of C, choose $R > 0$ such that $C_R(z_0)$ is contained in the interior of C. The function $\frac{1}{z-z_0}$ is analytic in $\mathbb{C}\setminus\{z_0\}$. Applying conclusion (3.7.1) of Theorem 3.7.2, we see that
$$\int_C \frac{1}{z-z_0}\,dz = \int_{C_R(z_0)} \frac{1}{z-z_0}\,dz = 2\pi i,$$