$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|. \tag{1.2.17}$$

The absolute value of the difference $z_1 - z_2$ *satisfies*

$$|z_1 - z_2| \leq |z_1| + |z_2|. \tag{1.2.18}$$

Moreover, we have the lower estimates

$$|z_1 + z_2| \geq \big||z_1| - |z_2|\big|, \tag{1.2.19}$$

and

$$|z_1 - z_2| \geq \big||z_1| - |z_2|\big|. \tag{1.2.20}$$

Proof. Consider a nondegenerate right triangle with vertices at 0, $z = (x, y)$, and $\operatorname{Re} z = x$, as shown in Figure 1.11. The sides of the triangle are $|\operatorname{Re} z| = |x|$, $|\operatorname{Im} z| = |y|$, and the hypotenuse is $|z|$. Since the hypotenuse is larger than either of the other two sides, we obtain (1.2.14). Since the sum of two sides in a triangle is larger than the third side, we obtain (1.2.15). Of course, (1.2.14) and (1.2.15) are also consequences of the inequalities

$$|x| \leq \sqrt{x^2 + y^2}, \quad |y| \leq \sqrt{x^2 + y^2}, \text{ and } \quad \sqrt{x^2 + y^2} \leq |x| + |y|,$$

which are straightforward to prove.

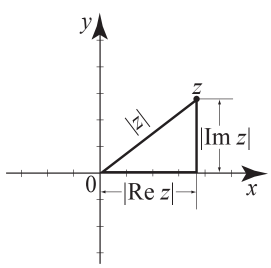

Fig. 1.11 Related to inequalities (1.2.14) and (1.2.15).

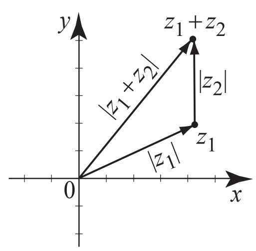

Fig. 1.12 Ineq. (1.2.16).

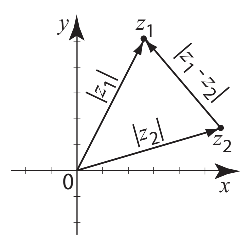

Fig. 1.13 Ineq. (1.2.20).

A geometric proof of the triangle inequality (1.2.16) is contained in Figure 1.12 where a triangle with sides $|z_1 + z_2|$, $|z_1|$, and $|z_2|$ appears. Then $|z_1 + z_2| \leq |z_1| + |z_2|$ is a consequence of the fact that the sum $|z_1| + |z_2|$ of two sides is larger than the length of the third side, which is $|z_1 + z_2|$. The triangle in Figure 1.13 with vertices at 0, z_1, and z_2 provides a geometric proof of the lower triangle inequality (1.2.20): The length $|z_1 - z_2|$ of the side of triangle is at least as big as the differences of the other two sides which are $|z_1| - |z_2|$ and $|z_2| - |z_1|$.

Since the triangle inequality (1.2.16) is fundamental in the development of complex analysis, we also offer an algebraic proof. Start by observing that

$$\overline{z_1 \overline{z_2}} = \overline{z_1}\,\overline{\overline{z_2}} = \overline{z_1}\, z_2.$$