$$\int_C f(z)\,dz = \sum_{j=1}^{n} \int_{C_j} f(z)\,dz. \tag{3.7.1}$$

Remark 3.7.3. We may informally say that in the hypothesis of the preceding theorem the function f is required to be analytic on a path[4] C and on its interior minus a few simply-connected "holes" denoted by Ω_j.

Proof. Fix a point z_0 on the outer path C. Join z_0 to a point w_1 in C_1 via a simple polygonal path L_1. Pick z_1 on C_1 and let P_1 be the part of C_1 from w_1 to z_1 traversed in the orientation of C_1. By Lemma 3.7.1, $\Omega \setminus (L_1 \cup P_1)$ is connected and thus there is a simple polygonal path L_2 disjoint from $L_1 \cup P_1$ that joins z_1 to a point w_2 in C_2.

Pick $z_2 \in C_2$ and let P_2 be the part of C_2 from w_2 to z_2 traversed in the orientation of C_2. By Lemma 3.7.1, $\Omega \setminus (L_1 \cup P_1 \cup L_2 \cup P_2)$ is connected and thus there is simple polygonal path L_3 disjoint from $L_1 \cup P_1 \cup L_2 \cup P_2$ that joins z_2 to a point w_3 in C_3.

Continuing in this fashion, we find points w_n and z_n on C_n and we let P_n be the part of C_n from w_n to z_n traversed in the same orientation. At the end, join z_n to a point w_{n+1} in C via a simple polygonal path L_{n+1} that does not intersect the previously selected path from z_0 to z_n, passing through $w_1, z_1, w_2, z_2, \dots, w_n$. In the selection of these points we defined

$$P_j = \text{part of } C_j \text{ from } w_j \text{ to } z_j \text{ traversed in the orientation of } C_j$$

for $j = 1, \dots, n$, and now also define

$$Q_j = \text{part of } C_j \text{ from } z_j \text{ to } w_j \text{ traversed in the orientation of } C_j.$$

Also let P be the part of C from z_0 to w_{n+1} and let Q be the part of C from w_{n+1} to z_0 both traversed in the orientation inherited by C.

This construction yields two simple closed paths Γ_1 and Γ_2, as illustrated in Figure 3.62, precisely defined as follows:

$$\Gamma_1 = \left[P, L_{n+1}^*, Q_n^*, L_n^*, Q_{n-1}^*, \dots, , Q_1^*, L_1^*\right]$$

$$\Gamma_2 = \left[L_1, P_1^*, L_2, P_2^*, L_3, \dots, P_n^*, L_{n+1}, Q\right]$$

for $j = 1, \dots, n$. (Recall that γ^* is the reverse of a path γ.)

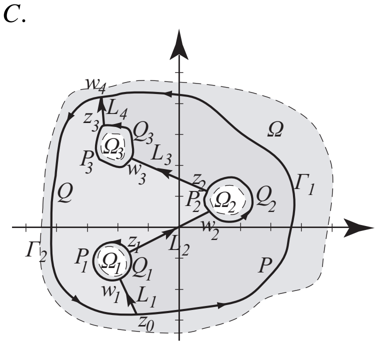

Fig. 3.62 The construction of Γ_1 and Γ_2.

Moreover, we have arranged so that all pieces of the complement of Ω in the interior of C do not lie in the interior of Γ_1 or Γ_2. Thus the interior regions of Γ_1 and Γ_2 are simply connected and f is analytic in a slightly larger simply connected

[4] analytic on C means analytic in a neighborhood of C