Integral Invariance for Homotopic Paths

Theorem 3.6.5. *Let f be an analytic function in a region Ω and let γ_0 and γ_1 be paths in Ω defined on $[0,1]$.*
(i) If $\gamma_0(0)=\gamma_1(0)=\alpha\in\Omega$, $\gamma_0(1)=\gamma_1(1)=\beta\in\Omega$, and γ_0 is homotopic to γ_1 according to Definition 3.6.1, then

$$\int_{\gamma_0} f(z)\,dz=\int_{\gamma_1} f(z)\,dz. \tag{3.6.9}$$

(ii) If γ_0 and γ_1 are closed homotopic paths then (3.6.9) holds.
(iii) If γ_0 is homotopic to a point in Ω, then

$$\int_{\gamma_0} f(z)\,dz=0. \tag{3.6.10}$$

Proof. Note that the assertion in (iii) follows from that in (ii) because the integral of a function over a point is zero. Also, the statement in (ii) is a consequence of that in (i). Indeed, let $\alpha=\gamma_0(0)=\gamma_0(1)$ and $\beta=\gamma_1(0)=\gamma_1(1)$. Define a path $\gamma_{\alpha,\beta}(s)=H(0,s)=H(1,s)$ in Ω that connects α to β. Then consider the path $\widetilde{\gamma}_1=\gamma_{\alpha,\beta}\cup\gamma_1\cup(\gamma_{\alpha,\beta})^*$, which starts and ends at the point α just like γ_0. Then $\widetilde{\gamma}_1$ is homotopic[2] to γ_0. Assuming the assertion in (i), (3.6.9) gives

$$\begin{aligned}\int_{\gamma_0} f(z)\,dz &= \int_{\widetilde{\gamma}_1} f(z)\,dz\\ &= \int_{\gamma_{\alpha,\beta}} f(z)\,dz+\int_{\gamma_1} f(z)\,dz+\int_{(\gamma_{\alpha,\beta})^*} f(z)\,dz\\ &= \int_{\gamma_1} f(z)\,dz,\end{aligned}$$

since $\int_{(\gamma_{\alpha,\beta})^*} f(z)\,dz=-\int_{\gamma_{\alpha,\beta}} f(z)\,dz$. Thus (ii) holds assuming (i).

We now prove the claimed assertion in (i). Let γ_0 and γ_1 be paths in Ω joining α to β. If Ω is the entire plane, which is a star-shaped region, Theorem 3.5.4, yields the claimed assertion. If Ω is not the entire plane, its boundary $\partial\Omega$ is closed and nonempty. Let H denote the mapping of the unit square $Q=[0,\,1]\times[0,1]$ into Ω that continuously deforms γ_0 into γ_1, and let $K=H[Q]$. Since Q is closed and bounded, and H is continuous, it follows that K is closed and bounded. Since K is contained in the (open region) Ω, it is disjoint from the closed boundary of Ω. So if δ denotes the distance from $\partial\Omega$ to K, then δ is positive (Appendix, page 483).

Since H is continuous on Q and Q is closed and bounded, it follows that H is uniformly continuous on Q. So for the given $\delta>0$ (which plays the role of $\varepsilon>0$ in the definition of uniform continuity), there is a positive integer n such that

[2] The mapping $\mathscr{H}(t,s)=H(0,3ts)$ for $0\le t\le\frac{1}{3}$, $\mathscr{H}(t,s)=H(3t-1,s)$ for $\frac{1}{3}\le t\le\frac{2}{3}$, $\mathscr{H}(t,s)=H(0,(3-3t)s)$ for $\frac{2}{3}\le t\le 1$, $0\le s\le 1$, continuously deforms γ_0 to $\widetilde{\gamma}_1$.