In other words, star-shaped sets contain all closed line segments starting from a fixed point z_0 and ending at an arbitrary point in the set. See Figure 3.39. Convex sets are always star-shaped about any point in them, but the converse is not true.

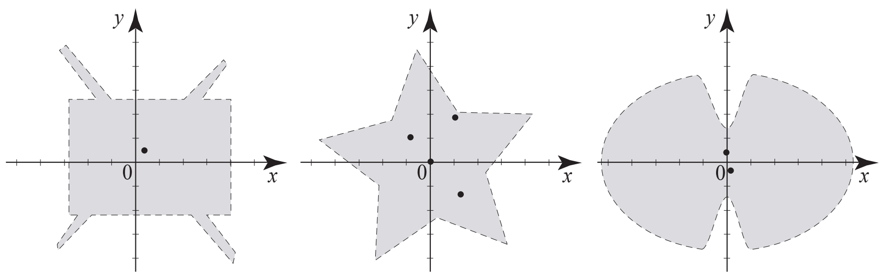

Fig. 3.39 These non-convex regions are star-shaped regions about the points indicated by dots.

Theorem 3.5.4. *Suppose that V is an open star-shaped subset of the plane and let f be an analytic function on V. Then for ~~a simple~~ any closed path γ contained in V we have*

$$\int_\gamma f(z)\,dz = 0\,. \tag{3.5.7}$$

Proof. Fix a point z_0 in V such that $[z_0,z]$ lies in V for all $z \in V$. Define the function

$$F(z) = \int_{[z_0,z]} f(\zeta)\,d\zeta, \qquad \text{for } z \in V.$$

Since the line segment is contained in V, the function F is well-defined. We show that F has a complex derivative equal to f. Let $z \in V$ and let $\delta > 0$ be such that $B_\delta(z)$ is contained in V. Then the closed triangle $[z_0,z,z+h,z_0]$ is contained in V and Theorem 3.5.2 yields

$$\int_{[z_0,z,z+h,z_0]} f(\zeta)\,d\zeta = 0.$$

In view of Properties (3.2.21) and (3.2.22) we have for $h \in B_\delta(0)$

$$\int_{[z_0,z]} f(\zeta)\,d\zeta + \int_{[z,z+h]} f(\zeta)\,d\zeta - \int_{[z_0,z+h]} f(\zeta)\,d\zeta = 0,$$

thus

$$\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_{[z,z+h]} f(\zeta)\,d\zeta\,.$$

Letting $h \to 0$ and using Lemma 3.3.3, we conclude that F has a complex derivative at any point z in V and $F'(z) = f(z)$. Then by Theorem 3.3.4, the integral of f around an arbitrary closed path γ contained in V is zero. ■