Continuing this process by induction, for each positive integer n there is a triangle $\triangle_n$ of size 4^{-n} of that of $\triangle$ such that

$$\triangle = \triangle_1 \supset \triangle_2 \supset \triangle_3 \supset \cdots \supset \triangle_n \supset \triangle_{n+1} \supset \cdots$$

with

$$\left| \int_{\partial\triangle_n} f(z)\,dz \right| \geq \frac{1}{4} \left| \int_{\partial\triangle_{n-1}} f(z)\,dz \right| \geq \cdots \geq \frac{1}{4^n} \left| \int_{\partial\triangle} f(z)\,dz \right|. \tag{3.5.6}$$

The collection $\{\triangle_n\}_{n=1}^{\infty}$ is a nested sequence of nonempty closed and bounded (compact) sets and by the nonempty intersection property (Appendix, page 484), there is a point z_0 in their intersection. This point is shown in Figure 3.38.

We now apply Lemma 3.5.1. Since f has a complex derivative at z_0, for the number $\varepsilon/(\mathrm{Diam}(\triangle))^2$ there is a $\delta > 0$ such that $B_\delta(z_0) \subset U$ and such that the integral of f over the boundary of any closed triangle containing z_0 and contained in $B_\delta(z_0)$ is at most $\varepsilon/(\mathrm{Diam}(\triangle))^2$. There is n large enough such that $\triangle_n$ is contained in $B_\delta(z_0)$; then

$$\left| \int_{\partial\triangle_n} f(z)\,dz \right| < \frac{\varepsilon}{\big(\mathrm{Diam}(\triangle)\big)^2} \big(\big(\mathrm{Diam}(\triangle_n)\big)\big)^2 .$$

But $\mathrm{Diam}(\triangle) = 2^n\,\mathrm{Diam}(\triangle_n)$, and thus it follows that

$$\left| \int_{\partial\triangle_n} f(z)\,dz \right| < \frac{\varepsilon}{4^n} .$$

Combining this fact with the inequality in (3.5.6), we obtain $\left| \int_{\partial\triangle} f(z)\,dz \right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, (3.5.5) holds. ■

Cauchy's Theorem for Star-shaped Domains

A subset V of the complex plane $\mathbb{C}$ is called **convex** if for all $a, b \in V$ we have

$$(1-t)a + tb \in V \qquad \text{for all } t \in [0,1].$$

Geometrically speaking, convex sets contain all closed line segments whose endpoints lie in the set. For instance, interiors of open or closed squares and disks are convex sets. The notion of star-shaped sets extends that of convex sets.

Definition 3.5.3. A subset V of the complex plane $\mathbb{C}$ is called **star-shaped** about a point z_0 in V if for all $z \in V$ we have

$$(1-t)z_0 + tz \in V \qquad \text{for all } t \in [0,1].$$