$$\int_\gamma -y\,dx + x\,dy = \iint_D 2\,dx\,dy = 2\times(\text{Area of } D).$$

For example, if γ is the unit circle with positive orientation, then the integral is equal to $2\times(\text{Area of unit disk}) = 2\pi$. □

We apply Green's theorem to evaluate path integrals of the form $\int_\gamma f(z)\,dz$ where γ is a simple closed path and f is an analytic function on γ and the region interior to γ. For this purpose, parametrize γ by $\gamma(t) = x(t) + i\,y(t)$, where $a \le t \le b$, and write $f(z) = u(x,y) + i\,v(x,y)$, where u and v are the real and imaginary parts of f, respectively. Then, by the definition of path integrals, we write

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_a^b \big[u(x(t),y(t)) + i\,v(x(t),y(t))\big]\big(x'(t) + i\,y'(t)\big)\,dt \\
&= \int_a^b \Big(u(x(t),y(t))x'(t) - v(x(t),y(t))y'(t)\Big)\,dt \\
&\qquad + i\int_a^b \Big(v(x(t),y(t))x'(t) + u(x(t),y(t))y'(t)\Big)\,dt \\
&= \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy)\,.
\end{aligned} \tag{3.4.2}$$

Thus the path integral $\int_\gamma f(z)\,dz$ is equal to a complex linear combination of two line integrals of real-valued functions. To remember (3.4.2), starting with $\int_\gamma f(z)\,dz$, we use $f(z) = u + iv$ and $dz = dx + i\,dy$ as follows:

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+idy) = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy).$$

We now state the main result of this section.

Theorem 3.4.4. (Cauchy's Theorem for Simple Paths) *Suppose that f is an analytic function on a region U that contains a simple closed path γ and its interior. If f' is continuous on U, then we have*

$$\int_\gamma f(z)\,dz = 0. \tag{3.4.3}$$

Remark 3.4.5. We later prove that derivatives of analytic functions are also analytic, hence continuous (Corollary 3.8.9). Thus the assumption that f' is continuous in Theorem 3.4.4 is superfluous. The merit of Theorem 3.4.4 lies in the fact that all analytic functions we encounter in examples have continuous derivatives.

Proof. Recall that if $f = u + iv$, u, v real-valued, then $f' = u_x + iv_x$ in view of (2.5.8), and the Cauchy-Riemann equations (2.5.7) hold: $u_x = v_y$ and $u_y = -v_x$. Using (3.4.2), and then applying Green's theorem along with the Cauchy-Riemann equations, we write