where $\gamma(z_0, z)$ is a path joining $z_0$ to $z$. Such a path exists since $\Omega$ is polygonally connected and the function in (3.3.5) is well defined, in view of the property that the integral is independent of path.

Since $\Omega$ is open and $z \in \Omega$, there is an open disc $B_r(z)$ centered at $z$ which is entirely contained in $\Omega$. For $z+w$ in $B_r(z)$, the line segment $[z, z+w]$ lies in $\Omega$.

Since the integral of $f$ is independent of path, we do not choose an arbitrary path that joins $z_0$ to $z+w$ but we choose the path

$$\gamma(z_0, z+w) = \big[\gamma(z_0, z), [z, z+w]\big],$$

formed by the path $\gamma(z_0, z)$ followed by the line segment $[z, z+w]$, as shown in Figure 3.25.

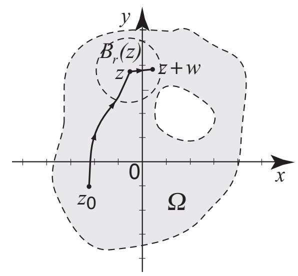


**Fig. 3.25** Picture of the proof.

Then, in view of Proposition 3.2.12 (iii), we have

$$\int_{\gamma(z_0, z+w)} f(\zeta)\, d\zeta = \int_{\gamma(z_0, z)} f(\zeta)\, d\zeta + \int_{[z, z+w]} f(\zeta)\, d\zeta$$

and so

$$F(z+w) - F(z) = \int_{\gamma(z_0, z+w)} f(\zeta)\, d\zeta - \int_{\gamma(z_0, z)} f(\zeta)\, d\zeta = \int_{[z, z+w]} f(\zeta)\, d\zeta .$$

Then,

$$\frac{F(z+w) - F(z)}{w} = \frac{1}{w} \int_{[z, z+w]} f(\zeta)\, d\zeta. \tag{3.3.6}$$

Taking the limit as $w \to 0$ and appealing to Lemma 3.3.3, we obtain that the function $F$ has a complex derivative at $z$ and $F'(z) = f(z)$.

Finally, it remains to prove the equivalence of $(b)$ and $(c)$. To see this, consider the closed path $\gamma$ in Figure 3.26 containing the point $z_1$.

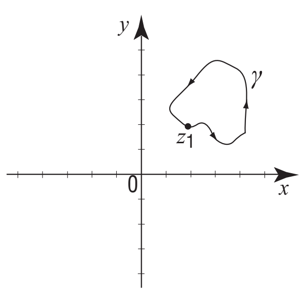


**Fig. 3.26** A closed path starting and ending at the same point $z_1$.

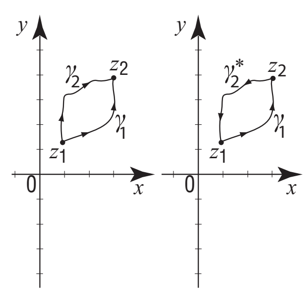


**Fig. 3.27** $\gamma_1$ followed by $\gamma_2^*$.