If $|w| < \delta$, we have $|tw| < \delta$ for all $t \in [0,1]$ and thus $|f(z+tw) - f(z)| < \varepsilon$. From the ML-inequality (3.2.31), we obtain for $|w| < \delta$

$$\left| \int_0^1 f(z+tw)\,dt - f(z) \right| = \left| \int_0^1 [f(z+tw) - f(z)]\,dt \right| \le \int_0^1 |f(z+tw) - f(z)|\,dt \le \varepsilon$$

and (3.3.1) follows using (3.3.2) by the (ε,δ)–definition of the limit. ■

We are now able to prove the main result of this section.

Theorem 3.3.4. (Independence of Path) *Let f be a continuous complex-valued function on a region Ω. Then the following assertions are equivalent:*
(a) There is an analytic function F on Ω such that $f(z) = F'(z)$ for all z in Ω.
(b) For arbitrary points z_1, z_2 and any path γ in Ω that joins z_1 to z_2, the integral

$$I = \int_\gamma f(z)\,dz$$

is independent of the path γ.
(c) The integral of f over all closed paths is zero.
Moreover, if (a) holds[1]*, then for any path γ in Ω that joins z_1 and z_2 we have*

$$\int_\gamma f(z)\,dz = F(z_2) - F(z_1). \tag{3.3.3}$$

Proof. If F is an antiderivative of f in Ω, then the complex-valued function $t \mapsto F(\gamma(t))$ is differentiable at the points t in (a,b) where $\gamma'(t)$ exists and we have

$$\frac{d}{dt}F(\gamma(t)) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t) \tag{3.3.4}$$

in view of Theorem 3.1.8. Now $t \mapsto f(\gamma(t))\gamma'(t)$ is piecewise continuous, because f is continuous and γ' is piecewise continuous. Also, since $F \circ \gamma$ is continuous, (3.3.4) tells us that $F \circ \gamma$ is a continuous antiderivative of $(f \circ \gamma)\,\gamma'$, in the sense of Theorem 3.2.7. Using this theorem, we deduce

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = F(\gamma(b)) - F(\gamma(a)) = F(z_2) - F(z_1),$$

completing the proof that (a) implies (b) and simultaneously deriving (3.3.3).

We now show that (b) implies (a). We only need to show that if I is independent of path, then f has an antiderivative F. Fix z_0 in Ω. For z in Ω, define

$$F(z) = \int_{\gamma(z_0,z)} f(\zeta)\,d\zeta, \tag{3.3.5}$$

[1] *and thus if (b) or (c) hold*