where α_k and β_k are in $[t_{k-1}, t_k]$. Then (3.2.28) becomes

$$\sum_{k=1}^{m} \sqrt{|x'(\alpha_k)|^2 + |y'(\beta_k)|^2}\,(t_k - t_{k-1}).$$

Recognizing this sum as a Riemann sum and taking limits as the partition gets finer, we recover the formula for arc length from calculus:

$$\ell(\gamma) = \int_a^b \sqrt{|x'(t)|^2 + |y'(t)|^2}\,dt = \int_a^b |\gamma'(t)|\,dt, \tag{3.2.29}$$

where the second equality follows from the complex notation $\gamma'(t) = x'(t) + i\,y'(t)$ and so $\sqrt{|x'(t)|^2 + |y'(t)|^2} = |\gamma'(t)|$.

For a piecewise smooth path γ, we repeat the preceding analysis for each smooth piece γ_j of γ and then add the lengths $\ell(\gamma_j)$'s. Definition 3.1.9 guarantees that each γ_j has a continuous derivative on the subinterval $[a_j, a_{j+1}]$ on which it is defined, thus $\ell(\gamma_j)$ is finite, hence so is $\ell(\gamma)$. This process yields formula (3.2.29) for the arc length of a piecewise smooth path γ as well, where the integrand in this case is piecewise continuous. The element of arc length is usually denoted by ds. Thus,

$$ds = \sqrt{|x'(t)|^2 + |y'(t)|^2}\,dt. \tag{3.2.30}$$

Example 3.2.18. (Arc length of cycloid) Let $a > 0$. Find the length of the arch of the cycloid $\gamma(t) = a(t - \sin t) + i\,a(1 - \cos t)$, where t ranges over the interval $[0, 2\pi]$.

The curve, illustrated in Figure 3.23, is formed by the trace of a fixed point on a moving circle that completes a full rotation.

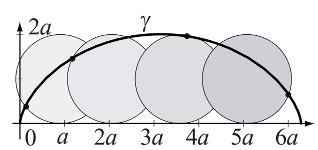

Fig. 3.23 First arch of the cycloid.

Solution. We have

$$\begin{aligned} x(t) &= a(t - \sin t) \quad &\Rightarrow \quad x'(t) &= a(1 - \cos t); \\ y(t) &= a(1 - \cos t) \quad &\Rightarrow \quad y'(t) &= a \sin t. \end{aligned}$$

Hence

$$\begin{aligned} ds &= \sqrt{x'(t)^2 + y'(t)^2}\,dt = \sqrt{a^2\Big((1 - \cos t)^2 + \sin^2 t\Big)}\,dt \\ &= a\sqrt{2(1 - \cos t)}\,dt = 2a \sin\left(\frac{t}{2}\right)\,dt. \end{aligned}$$

Applying (3.2.29), and using that $\sin(t/2) \geq 0$ for $0 \leq t \leq 2\pi$, we obtain the length of the arch