(iii) The path Γ is defined piecewise on the interval $[0,m]$ by

$$\Gamma(t) = \gamma_k(t) \text{ for } k-1 \le t \le k. \tag{3.2.23}$$

Then we have

$$\begin{aligned}\int_\Gamma f(z)dz &= \int_0^m f(\Gamma(t))\Gamma'(t)\,dt = \sum_{k=1}^m \int_{k-1}^k f(\Gamma(t))\Gamma'(t)\,dt \\ &= \sum_{k=1}^m \int_k^{k+1} f(\gamma_k(\tau))\gamma_k'(\tau)\,d\tau = \sum_{k=1}^m \int_{\gamma_k} f(z)\,dz.\end{aligned}$$

Thus we obtain (3.2.22). ■

Here is an illustration that uses the linearity of the path integral (3.2.20) and the familiar relations

$$x = \frac{z+\bar z}{2} \qquad \text{and} \qquad y = \frac{z-\bar z}{2i}.$$

Example 3.2.13. Let $C_1(0)$ denote the positively oriented unit circle traced once. Compute

$$\int_{C_1(0)} x\,dz = \int_{C_1(0)} \operatorname{Re} z\,dz.$$

Solution. Using (3.2.18) and (3.2.19), we have

$$\int_{C_1(0)} x\,dz = \int_{C_1(0)} \frac{z+\bar z}{2}\,dz = \frac{1}{2}\underbrace{\int_{C_1(0)} z\,dz}_{0} + \frac{1}{2}\underbrace{\int_{C_1(0)} \bar z\,dz}_{2\pi i} = \pi i. \qquad \square$$

The integrals in previous examples involved smooth curves. In the following example, we compute integrals over polygonal paths, which are piecewise smooth.

Example 3.2.14. (Integrals over polygonal paths) Let $z_1 = -1$, $z_2 = 1$, and $z_3 = i$ (see Figure 3.21). Compute

(a) $\displaystyle\int_{[z_1,z_2]} \bar z\,dz$ (b) $\displaystyle\int_{[z_2,z_3]} \bar z\,dz$ (c) $\displaystyle\int_{[z_3,z_1]} \bar z\,dz$

(d) $\displaystyle\int_{[z_1,z_2,z_3]} \bar z\,dz$ (e) $\displaystyle\int_{[z_1,z_2,z_3,z_1]} \bar z\,dz$ (f) $\displaystyle\int_{[z_1,z_3]} \bar z\,dz.$