Not all properties of the derivative of a real-valued function hold for complex-valued functions. Most notably, the mean value property fails for complex-valued functions. For real-variable functions, the mean value property states that if f is continuous on $[a, b]$ and differentiable on (a, b), then $f(b) - f(a) = f'(c)(b-a)$ for some c in (a, b).

This property does not hold for complex-valued functions. To see this, consider $f(x) = e^{ix}$ for x in $[0, 2\pi]$. Then f is continuous on $[0, 2\pi]$ and has a derivative $f'(x) = i\,e^{ix}$ on $(0, 2\pi)$. Also,

$$f(2\pi) - f(0) = 1 - 1 = 0,$$

but f' never vanishes since

$$|f'(x)| = |i\,e^{ix}| = |e^{ix}| = 1.$$

Hence there is no number c in $(0, 2\pi)$ such that $f(2\pi) - f(0) = (2\pi - 0)f'(c)$, and so the mean value property does not hold for complex-valued functions (Figure 3.9).

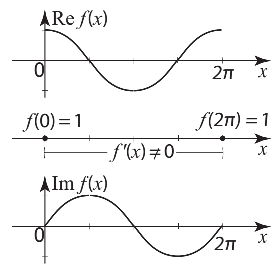

Fig. 3.9 The mean value property fails for complex-valued functions of a real variable.

Paths (Contours)

We introduce the notion of a path, which is fundamental in the theory of complex integration. Recall that a curve already contains intrinsically the notion of continuity. We now attach to it the fundamental analytical property of differentiability.

A continuous map f from $[a,b]$ to $\mathbb{C}$ is called **continuously differentiable** if it is also differentiable on $[a,b]$ and its derivative f' is continuous on $[a,b]$. The derivative f' is defined at a and b as the one-sided limits $f'(a) = \lim_{t\downarrow a} \frac{f(t)-f(a)}{t-a}$ and $f'(b) = \lim_{t\uparrow b} \frac{f(t)-f(b)}{t-b}$. We extend this notion by allowing γ to have one-sided derivatives at finitely many points in (a,b).

Definition 3.1.9. A continuous complex-valued function $f(t)$ defined on a closed interval $[a,b]$ is called **piecewise continuously differentiable** if there exist points $a_1 < a_2 < \cdots < a_{m-1}$ in (a,b) such that f' is continuously differentiable on each interval $[a_j, a_{j+1}]$ for $j = 0, 1, \ldots, m-1$, where $a_0 = a$ and $a_m = b$. In other words: (i) $f'(t)$ exists for all t in (a_j, a_{j+1}) and at the endpoints a_j, a_{j+1} as one-sided limit. (ii) f' is continuous on each interval $[a_{j-1}, a_j]$ for $j = 1, \ldots, m$.

Note that $f' : [a,b] \to \mathbb{C}$ may be discontinuous at some a_j.