Chapter 3
Complex Integration

Nature laughs at the difficulties of integration.

-Pierre-Simon de Laplace (1749-1827)

In this chapter we study integrals of complex-variable functions over paths. Paths are piecewise continuously differentiable maps from closed intervals to the complex plane. An important result proved in this chapter is an analog of the Fundamental Theorem of Calculus for continuous functions with complex antiderivatives. This analog says that the integral of the derivative of an analytic function is equal to the difference of the values of the function at the endpoints. We also investigate an important question concerning the dependence of an integral on the path of integration.

Cauchy's theorem states that the integral over a simple closed path of an analytic function defined on an open set that contains the path and its interior must be zero. The first version (Section 3.4) is sufficient for the development of the course; this is based on Green's theorem from advanced calculus and makes the assumption that derivatives of analytic functions are continuous. In the following three sections, we prove versions of Cauchy's theorem without the continuity assumption on the derivatives. These versions involve theoretical notions, such as deformation of paths and simple connectedness, and offer geometric intuition and many applications. These sections may be omitted without interrupting the flow of the course.

In Section 3.8 we derive Cauchy's generalized integral formula, which facilitates the computation of several integrals and provides the basis for many applications. We illustrate the power of this formula by providing a simple proof of the fundamental theorem of algebra and by deriving various striking properties of analytic functions, including the mean value property and the maximum modulus principle.

3.1 Paths (Contours) in the Complex Plane

In this text a **curve** is defined as ~~the graph of a continuous function $y = f(x)$.~~ a continuous function from an interval $[a,b]$ to the complex plane, but it is often understood as the image of this mapping. ~~This~~ A curve could be written in parametric form by expressing x and y as functions of a third variable t. For example, the semicircle $y = \sqrt{1-x^2}$, $-1 \le x \le 1$, in Figure 3.1 could be parametrized by the equations

$$x = x(t) = \cos t, \qquad y = y(t) = \sin t, \quad 0 \le t \le \pi.$$