*Proof.* We only consider the case $\operatorname{Re} f = \operatorname{Re} g$ on $\Omega$ since the case $\operatorname{Im} f = \operatorname{Im} g$ is almost identical and is left to Exercise 32. Let $h = f - g = u + iv$, with $u, v$ real-valued. We want to show that $h = c$ on $\Omega$. Since $h$ is analytic, it is enough by Theorem 2.5.7 to show that $h' = 0$ on $\Omega$. We have $u = \operatorname{Re} h = \operatorname{Re} f - \operatorname{Re} g = 0$ on $\Omega$, and so $u_x = u_y = 0$ on $\Omega$. By the Cauchy-Riemann equations, $v_x = -u_y = 0$. Consequently, by (2.5.8), $h' = u_x + iv_x = 0$ on $\Omega$. ■

# Exercises 2.5

*In Exercises* 1–14*, use Corollary 2.5.2 to determine the set on which the functions are analytic and compute their complex derivatives using either equation in (2.5.8).*

**1.** $z$ **2.** $z^2$ **3.** $e^{z^2}$

**4.** $2x + 3iy$ **5.** $e^{\bar{z}}$ **6.** $\dfrac{y - ix}{x^2 + y^2}$

**7.** $\dfrac{1}{z+1}$ **8.** $z^3 - 2z$ **9.** $ze^z$

**10.** $\cos z$ **11.** $\sin(2z)$ **12.** $\cosh z$

**13.** $|z|^2$ **14.** $\dfrac{x^4 + i2xy(x^2 + y^2) - y^4 + x - iy}{x^2 + y^2}$

*In Exercises* 15–26*, use properties of the derivative to compute the complex derivatives of the functions and determine the largest set on which they are analytic. In Exercises* $23 - 26$*, use the principal branch of the power.*

**15.** $ze^{z^2}$ **16.** $(1 + e^z)^5$ **17.** $\sin z \cos z$

**18.** $\operatorname{Log}(z+1)$ **19.** $\dfrac{\operatorname{Log}(3z-1)}{z^2+1}$ **20.** $\sinh(3z + i)$

**21.** $\cosh(z^2 + 3i)$ **22.** $\log_{\frac{\pi}{2}}(z+1)$ **23.** $z^i$

**24.** $(z+1)^{1/2}$ **25.** $\dfrac{1}{(z-i)^{1/2}}$ **26.** $z^z$

*Solve Exercises* 27 *and* 28 *by identifying the limit as a complex derivative; Solve Exercises* 29 *and* 30 *using L'Hospital's rule (Exercise* 24*, Section* 2.3*).*

**27.** $\displaystyle\lim_{z\to 0} \frac{\sin z}{z}$ **28.** $\displaystyle\lim_{z\to 0} \frac{e^z - 1}{z}$

**29.** $\displaystyle\lim_{z\to 0} \frac{\operatorname{Log}(z+1)}{z}$ **30.** $\displaystyle\lim_{z\to i} \frac{1 + iz}{z(z-i)}$

**31.** Define the **principal branch** of the inverse tangent by taking the principal branch of the logarithm as in (1.8.12):

$$\tan^{-1} z = \frac{i}{2} \operatorname{Log}\left(\frac{1 - iz}{1 + iz}\right).$$

Compute the derivative of $\tan^{-1} z$.

**32.** Complete the proof of Corollary 2.5.8 by treating the case $\operatorname{Im} f = \operatorname{Im} g$ on $\Omega$.

**33.** Suppose that $f = u + iv$ is analytic in a region $\Omega$. Show that ~~that~~ if either $\operatorname{Re} f$ or $\operatorname{Im} f$ are constant on $\Omega$, then $f$ must be constant on $\Omega$.

**34.** Suppose that $f = u + iv$ is analytic in a region $\Omega$. Show that
(a) $f' = u_x - iu_y$ and $f' = v_y + iv_x$;
(b) $|f'|^2 = u_x^2 + u_y^2 = v_x^2 + v_y^2$.

**35.** Suppose that $f$ and $\overline{f}$ are analytic in a region $\Omega$. Show that $f$ must be constant in $\Omega$. [Hint: Consider $f + \overline{f}$ and use Exercise 33.]