Note that
$$|\varepsilon_1(x,y)| \le |\varepsilon(x,y)|, \qquad |\varepsilon_2(x,y)| \le |\varepsilon(x,y)|$$
and since $\varepsilon(x,y)$ tends to zero as $(x,y) \to (x_0,y_0)$, both $\varepsilon_1(x,y)$ and $\varepsilon_2(x,y)$ tend to zero as $(x,y) \to (x_0,y_0)$. Hence both u and v are differentiable at (x_0,y_0). Consequently, the partial derivatives of u and v exist (Theorem 2.4.3) and we have

$$\begin{aligned} u_x(x_0,y_0) &= A \qquad & u_y(x_0,y_0) &= -B \\ v_x(x_0,y_0) &= B \qquad & v_y(x_0,y_0) &= A. \end{aligned}$$

The Cauchy-Riemann equations (2.5.7) follow from this.

To prove the converse direction we assume that u and v are differentiable on U and satisfy (2.5.7). By the definition of differentiability (Definition 2.4.1) we write

$$\begin{aligned} u(x,y)-u(x_0,y_0) &= u_x(x_0,y_0)(x-x_0)+u_y(x_0,y_0)(y-y_0)+\varepsilon_1(x,y)|(x-x_0,y-y_0)| \\ v(x,y)-v(x_0,y_0) &= v_x(x_0,y_0)(x-x_0)+v_y(x_0,y_0)(y-y_0)+\varepsilon_2(x,y)|(x-x_0,y-y_0)| \end{aligned}$$

where ~~the~~ $\varepsilon_1(x,y)$, $\varepsilon_2(x,y)$ are functions that tend to zero as $(x,y) \to (x_0,y_0)$. Adding the displayed expressions (after multiplying the second one by i), we obtain

$$\begin{aligned} & f(x+iy)-f(x_0+iy_0) \\ &= (u_x(x_0,y_0)+iv_x(x_0,y_0))(x-x_0)+(u_y(x_0,y_0)+i\,v_y(x_0,y_0))(y-y_0) \\ &\qquad\qquad +\varepsilon_1(x,y)|(x-x_0,y-y_0)|+i\,\varepsilon_2(x,y)|(x-x_0,y-y_0)| \\ &= (u_x(x_0,y_0)+iv_x(x_0,y_0))(x-x_0)+(-v_x(x_0,y_0)+i\,u_x(x_0,y_0))(y-y_0) \\ &\qquad\qquad +\varepsilon_1(x,y)|(x-x_0,y-y_0)|+i\,\varepsilon_2(x,y)|(x-x_0,y-y_0)| \\ &= (u_x(x_0,y_0)+iv_x(x_0,y_0))(x-x_0+i(y-y_0))+E(x+iy)\big(x-x_0+i(y-y_0)\big) \end{aligned}$$

where in the second equality we used assumption (2.5.7) and we set

$$E(x+iy) = \varepsilon_1(x,y)\frac{|(x-x_0,y-y_0)|}{x-x_0+i(y-y_0)} + i\,\varepsilon_2(x,y)\frac{|(x-x_0,y-y_0)|}{x-x_0+i(y-y_0)}\,.$$

Notice that
$$|E(x+iy)| \le |\varepsilon_1(x,y)|+|\varepsilon_2(x,y)|\,,$$
which tends to 0 as $(x,y) \to (x_0,y_0)$. We have now shown that

$$f(z)-f(z_0) = (u_x(z_0)+iv_x(z_0))(z-z_0)+E(z)(z-z_0)$$

which implies that f is analytic and that $f'(z_0) = u_x(z_0)+iv_x(z_0)$. This proves one identity in (2.5.8), while the other one is a consequence of this one and (2.5.7). ■

Corollary 2.5.2. *If u,v are real-valued functions defined on an open subset U of $\mathbb{R}^2$ which have continuous partial derivatives that satisfy $u_x = v_y$, $u_y = -v_x$, then the complex-valued function $f(x+iy) = u(x,y)+iv(x,y)$ is analytic on U.*

Proof. Apply Theorems 2.4.4 and 2.5.1. ■