where in the last step we have used $1/i=-i$ and rearranged the terms. Recognizing the partial derivatives of v and u with respect to y we obtain

$$f'(z_0)=\frac{\partial v}{\partial y}(x_0,y_0)-i\,\frac{\partial u}{\partial y}(x_0,y_0), \tag{2.5.5}$$

which is this time an expression of the derivative of f in terms of the partial derivatives with respect to y of u and v. Equating real and imaginary parts in (2.5.3) and (2.5.5), we deduce the following equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}. \tag{2.5.6}$$

These are called the **Cauchy-Riemann equations**. They first appeared in 1821 in the early work of Cauchy on integrals of complex-valued functions. Their connection to the existence of the complex derivative, appeared in 1851 in the doctoral dissertation of the German mathematician Bernhard Riemann (1826–1866).

Theorem 2.5.1. (Cauchy-Riemann Equations) *Let U be an open subset of $\mathbb{R}^2$ and let u, v be real-valued functions defined on U. Then the complex-valued function $f(x+iy)=u(x,y)+iv(x,y)$ is analytic on U if and only if u,v are differentiable functions on U and satisfy the Cauchy-Riemann equations*

$$u_x=v_y \qquad \textit{and} \qquad u_y=-v_x \tag{2.5.7}$$

for all points in U. If this is the case, then for all $(x,y)\in U$ we have

$$f'(x+iy)=u_x(x,y)+i\,v_x(x,y) \quad \textit{or} \quad f'(x+iy)=v_y(x,y)-i\,u_y(x,y). \tag{2.5.8}$$

Proof. Let us use the notation $z=x+iy$ for a general point in U and $z_0=x_0+iy_0$ for a fixed point in U, where x,y,x_0,y_0 are real numbers. In view of Proposition 2.3.10, if the function f is analytic, we have

$$f(z)-f(z_0)=f'(z_0)(z-z_0)+\varepsilon(z)(z-z_0) \tag{2.5.9}$$

where $\varepsilon(z)$ ends to zero as $z\to z_0$. Setting $f'(z_0)=A+iB$, where A,B are real, and splitting up real and imaginary parts in (2.5.9) we obtain

$$\begin{aligned} u(x,y)-u(x_0,y_0)&=A(x-x_0)-B(y-y_0)+\varepsilon_1(x,y)|(x-x_0,y-y_0)|\\ v(x,y)-v(x_0,y_0)&=B(x-x_0)+A(y-y_0)+\varepsilon_2(x,y)|(x-x_0,y-y_0)|, \end{aligned}$$

where

$$\begin{aligned} \varepsilon_1(x,y)&=\mathrm{Re}\left[\varepsilon(x+iy)\frac{x-x_0+i(y-y_0)}{|(x-x_0,y-y_0)|}\right]\\ \varepsilon_2(x,y)&=\mathrm{Im}\left[\varepsilon(x+iy)\frac{x-x_0+i(y-y_0)}{|(x-x_0,y-y_0)|}\right]. \end{aligned}$$