The limit in (2.4.3) involves the values of u at the point $(x+h, y)$. This point belongs to Ω if h is sufficiently small, because Ω is open. It is in this sense that we interpret expressions involving limits.

Theorem 2.4.3. *Suppose u is differentiable at (x_0, y_0), so that (2.4.2) holds. Then*
(i) u is continuous at (x_0, y_0); and
(ii) u_x, u_y exist at (x_0, y_0) and $u_x(x_0, y_0) = A$, $u_y(x_0, y_0) = B$.

Proof. Taking limits on both sides of (2.4.2) as $(x, y) \to (x_0, y_0)$, we obtain

$$u(x,y) = u(x_0,y_0) + \overbrace{A(x-x_0)}^{\to 0} + \overbrace{B(y-y_0)}^{\to 0} + \overbrace{\varepsilon(x,y)|(x-x_0, y-y_0)|}^{\to 0}$$

and it follows that $u(x,y) \to u(x_0,y_0)$. Hence, u is continuous at (x_0,y_0), and (i) is proved. For (ii), we only prove that $u_x(x_0, y_0) = A$, the second part being similar. To compute $u_x(x_0, y_0)$, we fix $y = y_0$ and take the derivative of $u(x, y_0)$ with respect to x. From (2.4.2) we have

$$u_x(x_0, y_0) = \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} = A + \lim_{x \to x_0} \varepsilon(x, y_0) \frac{|x - x_0|}{x - x_0} = A,$$

since ~~the~~ the second limit is zero; this follows from the fact that $\left|\varepsilon(x, y_0)\frac{|x-x_0|}{x-x_0}\right| = |\varepsilon(x, y_0)| \to 0$ as $(x, y) \to (x_0, y_0)$. ■

The converse of part (ii) of Theorem 2.4.3 is not true. A function of two variables may have partial derivatives and yet fail to be differentiable at a point. In fact, the function may not even be continuous at that point. As an illustration, consider the function of two variables

$$u(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{when } (x,y) \neq (0,0), \\ 0 & \text{when } (x,y) = (0,0). \end{cases} \tag{2.4.5}$$

In Exercise 1, you are asked to verify that

$$u_x(0,0) = u_y(0,0) = 0,$$

and that u is not continuous at $(0, 0)$. Hence by Theorem 2.4.3(i), the function u is not differentiable at the point $(0, 0)$. The graph of the function u is shown in the Figure 2.17.

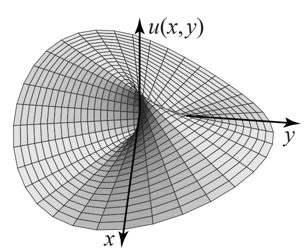

Fig. 2.17 The function $u(x,y)$ in (2.4.5).

To obtain differentiability at a point, more than the existence of the partial derivatives is needed. We have the following interesting result.