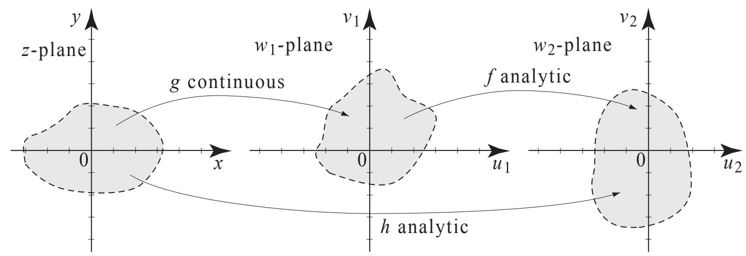

Fig. 2.14 In the reverse chain rule we suppose that g is continuous and that $h = f \circ g$ is analytic and we conclude that g is analytic.

Theorem 2.3.12. (Reverse Chain Rule) *Suppose that g is a continuous function on a region Ω and f is an analytic function on a region U that contains $g[\Omega]$. Suppose that $h = f \circ g$ is analytic on Ω and that $f'(g(z)) \neq 0$ for all z in Ω. Then g is analytic on Ω and*

$$g'(z) = \frac{h'(z)}{f'(g(z))}, \qquad z \in \Omega. \tag{2.3.18}$$

Proof. Let z_0 be in Ω. We know that $h(z) = f(g(z))$ is analytic at $z = z_0$, f is analytic at $g(z_0)$ with $f'(g(z_0)) \neq 0$, and g is continuous at z_0. We want to show that

$$g'(z_0) = \frac{h'(z_0)}{f'(g(z_0))}. \tag{2.3.19}$$

Applying Proposition 2.3.10 to $h(z) = f(g(z))$, we write

$$f(g(z)) = f(g(z_0)) + h'(z_0)(z - z_0) + \varepsilon(z)(z - z_0), \quad \varepsilon(z) \to 0 \text{ as } z \to z_0. \tag{2.3.20}$$

Applying Proposition 2.3.10 to f at $g(z_0)$, we have

$$f(g(z)) = f(g(z_0)) + f'(g(z_0))(g(z) - g(z_0)) + \eta(g(z))(g(z) - g(z_0)), \tag{2.3.21}$$

where $\eta(g(z)) \to 0$ as $g(z) \to g(z_0)$ or, equivalently, as $z \to z_0$ by continuity of g at z_0. Subtract (2.3.21) from (2.3.20) and rearrange the terms to get

$$\frac{g(z) - g(z_0)}{z - z_0} = \frac{h'(z_0) + \varepsilon(z)}{f'(g(z_0)) + \eta(g(z))}. \tag{2.3.22}$$

As $z \to z_0$, $\varepsilon(z) \to 0$ and $\eta(g(z)) \to 0$, implying (2.3.19). Notice that the denominator in (2.3.22) does not vanish for z sufficiently close to z_0. ∎