$$\frac{f(z)-f(z_0)}{z-z_0} = A + \varepsilon(z). \tag{2.3.13}$$

Taking the limit as $z \to z_0$ and using the fact that $\varepsilon(z) \to 0$, we conclude that $f'(z_0)$ exists and equals A. ■

So far we have been successful in differentiating polynomials and rational functions. To go beyond these examples we need more tools, such as composition of functions. The formalism of Proposition 2.3.10 greatly simplifies the proofs related to compositions of analytic functions.

Theorem 2.3.11. (Chain Rule) *Suppose that g is analytic on an open set U and that f is analytic ~~and~~ on an open set containing $g[U]$. Then $f \circ g$ is an analytic function on U. Moreover, for z_0 in U the chain rule identity holds*

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0). \tag{2.3.14}$$

Proof. Suppose g is analytic at z_0 and f is analytic at $g(z_0)$. We want to show that

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0). \tag{2.3.15}$$

Since g is analytic at z_0, appealing to Proposition 2.3.10, we write

$$\frac{g(z)-g(z_0)}{z-z_0} = g'(z_0) + \varepsilon(z), \qquad \varepsilon(z) \to 0 \text{ as } z \to z_0. \tag{2.3.16}$$

Also, f is analytic at $g(z_0)$, by Proposition 2.3.10 again, we write

$$f(w) - f(g(z_0)) = f'(g(z_0))(w - g(z_0)) + \eta(w)(w - g(z_0)),$$

where $\eta(w) \to 0$ as $w \to g(z_0)$.

Replacing w by $g(z)$, dividing by $z - z_0$, and using (2.3.16), we obtain

$$\frac{f(g(z)) - f(g(z_0))}{z - z_0} = f'(g(z_0))\big(g'(z_0) + \varepsilon(z)\big) + \eta(g(z))\big(g'(z_0) + \varepsilon(z)\big). \tag{2.3.17}$$

As $z \to z_0$, $\varepsilon(z) \to 0$, $g(z) \to g(z_0)$ by continuity, and so $\eta(g(z)) \to 0$. Using this in (2.3.17), we conclude that

$$\lim_{z \to z_0} \frac{f(g(z)) - f(g(z_0))}{z - z_0} = f'(g(z_0))g'(z_0),$$

as asserted by the chain rule. ■

The following inside-out chain rule (illustrated in Figure 2.14) is useful when dealing with inverse functions such as logarithms and powers.