48. Find the two square roots of i.

49. Find the two square roots of $-3+4i$.

50. Project Problem: The cubic equation. We derive the solution of the cubic equation

$$x^3+ax^2+bx+c=0, \tag{1.1.8}$$

where $a, b,$ and c are real numbers.
(a) Use the change of variables $x=y-\frac{a}{3}$ to transform the equation to the following reduced form

$$y^3+py+q=0, \tag{1.1.9}$$

which does not contain a quadratic term in y, where $p=b-\frac{a^2}{3}$ and $q=\frac{2a^3}{27}-\frac{ab}{3}+c$. (This trick is due to the Italian mathematician Niccolò Tartaglia (1500–1557).)
(b) Let $y=u+v$, and show that $u^3+v^3+(3uv+p)(u+v)+q=0$.
(c) Require that $3uv+p$=0; then directly we have $u^3v^3=-\frac{p^3}{27}$, and from the equation in part (b) we have $u^3+v^3=-q$.
(d) Suppose that U and V are numbers satisfying $U+V=-\beta$ and $UV=\gamma$. Show that U and V are solutions of the quadratic equation $X^2+\beta X+\gamma=0$.
(e) Use (c) and (d) to conclude that u^3 and v^3 are solutions of the quadratic equation $X^2+qX-\frac{p^3}{27}=0$. Thus,

$$u=\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}} \quad \text{and} \quad v=\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}.$$

(f) Derive a solution of (1.1.8),

$$x=\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}-\frac{a}{3}.$$

This is **Cardan's formula**, named after him because he was the first one to publish it. In the case $\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3\geq 0$, the formula clearly yields one real root of (1.1.8). You can use this root to factor (1.1.8) down into a quadratic equation, which you can solve to find all the roots of (1.1.8). The case $\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3<0$ baffled the mathematicians of the sixteenth century. They knew that the cubic equation (1.1.8) must have at least one real root, yet the solution in this case involves square roots of negative numbers, which are imaginary numbers. It turns out in this case that u and v are complex conjugate numbers, hence their sum is a real number and the solution x is real! This was discovered by the Italian mathematician Rafael Bombelli (1527–1572) (see Exercise 51). Not only was Bombelli bold enough to work with complex numbers; by using them to generate real solutions, he demonstrated that complex numbers were not merely the product of our imagination but tools that are essential to derive real solutions. This theme will occur over and over again in this book when we will appeal to complex variable techniques to solve real-life problems calling for real-valued solutions.

For an interesting account of the history of complex numbers, we refer to the book *The History of Mathematics, An Introduction*, 3rd edition, by David M. Burton (McGraw-Hill, 1997).

51. (Bombelli's equation) An equation of historical interest is $x^3-15x-4=0$, which was investigated by Bombelli.
(a) Use Cardan's formula to derive the solution

$$x=u+v=\sqrt[3]{2+11i}+\sqrt[3]{2-11i},$$

where u is the first cube root and v is the second.
(b) Bombelli had the incredible insight that u and v have to be conjugate for $u+v$ to be real. Set $u=a+ib$ and $v=a-ib$, where a and b are to be determined. Cube both sides of the equations