Since the limit along C is not equal to the limit along C', we conclude that the limit in (2.3.9) does not exist. Hence the function $\bar{z}$ is not analytic at z_0. Since z_0 is arbitrary, it follows that $\bar{z}$ is nowhere analytic.
(b) We follow the approach in (a) and use the same directions along C and C'. For z on C, $\operatorname{Re} z - \operatorname{Re} z_0 = x_0 + t - x_0 = t$, and, for z on C', $\operatorname{Re} z - \operatorname{Re} z_0 = x_0 - x_0 = 0$.

Thus we write

$$\lim_{\substack{z \to z_0 \\ z \text{ on } C}} \frac{\operatorname{Re} z - \operatorname{Re} z_0}{z - z_0} = \lim_{t \to 0} \frac{t}{t} = 1$$

and

$$\lim_{\substack{z \to z_0 \\ z \text{ on } C'}} \frac{\operatorname{Re} z - \operatorname{Re} z_0}{z - z_0} = \lim_{t \to 0} \frac{0}{it} = 0.$$

So the derivative of $\operatorname{Re} z$ does not exist at z_0. Since z_0 is arbitrary, we conclude that $\operatorname{Re} z$ is nowhere analytic.

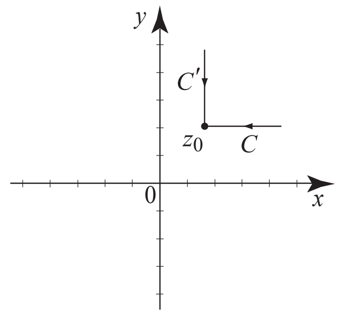

Fig. 2.13 For $z \in C$ we have $z - z_0 = t$ while for $z \in C'$, $z - z_0 = it$.

There is also a quick proof of (b) based on the result of (a) and the identity $\bar{z} = 2\operatorname{Re} z - z$. In fact, if $\operatorname{Re} z$ has a derivative at z_0, then by the properties of the derivative it would follow that $\bar{z}$ has a derivative at z_0, which contradicts (a). □

Suppose that $f(z)$ has a complex derivative at a point z_0 and let

$$\varepsilon(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0). \tag{2.3.10}$$

Then $\varepsilon(z) \to 0$ as $z \to z_0$, because the difference quotient in (2.3.10) tends to $f'(z_0)$. Solving for $f(z)$ in (2.3.10) we obtain

$$f(z) = \overbrace{f(z_0) + f'(z_0)(z - z_0)}^{\text{linear function of } z} + \varepsilon(z)(z - z_0). \tag{2.3.11}$$

This expression shows that, near a point where f ~~is analytic~~has a complex derivative, $f(z)$ is approximately a linear function. The converse is also true.

Proposition 2.3.10. *Let U be an open subset of $\mathbb{C}$. A function f on U has a complex derivative at a point $z_0 \in U$ if and only if there is a complex number A and a function $\varepsilon(z)$ such that*

$$f(z) = f(z_0) + A(z - z_0) + \varepsilon(z)(z - z_0), \tag{2.3.12}$$

and $\varepsilon(z) \to 0$ as $z \to z_0$. If this is the case, then $A = f'(z_0)$.

Proof. We have already one direction. For the other direction, suppose that $f(z)$ can be written as in (2.3.12). Then, for $z \neq z_0$,