For $z = iy$ with y real, we have $\operatorname{Re} z = 0$, and so

$$\lim_{z\to 0} \frac{\operatorname{Re} z}{z} = \lim_{iy\to 0} \frac{0}{iy} = 0.$$

Since we have obtained different limits as we approached 0 in different ways, we conclude that the function $\frac{\operatorname{Re} z}{z}$ has no limit as $z \to 0$. $\square$

The next example involves a function with infinitely many nonremovable discontinuities.

Example 2.2.17. (The nonremovable discontinuities of $\operatorname{Arg} z$**)** The principal branch of the argument $\operatorname{Arg} z$ takes the value of argument z that is in the interval $-\pi < \operatorname{Arg} z \le \pi$. It is not defined at $z = 0$ and hence $\operatorname{Arg} z$ is not continuous at $z = 0$. We show that $z = 0$ is not a removable discontinuity of $\operatorname{Arg} z$ by showing that $\lim_{z\to 0} \operatorname{Arg} z$ does not exist.

Indeed, if $z = x > 0$, then $\operatorname{Arg} z = 0$ and so $\lim_{z=x\downarrow 0} \operatorname{Arg} z = 0$, where the down-arrow denotes the limit from the right, also denoted as $\lim_{z=x\to 0^+} \operatorname{Arg} z$. However, if $z = x < 0$, then $\operatorname{Arg} z = \pi$ and so $\lim_{z=x\uparrow 0} \operatorname{Arg} z = \pi$, where the up-arrow denotes the limit from the left, also denoted as $\lim_{z=x\to 0^-} \operatorname{Arg} z$. By the uniqueness of limits, we conclude that $\lim_{z\to 0} \operatorname{Arg} z$ does not exist. Also, for a point on the negative x-axis, $z_0 = x_0 < 0$, we have $\operatorname{Arg} z_0 = \pi$. If z approaches z_0 from the second quadrant, say along a curve C as in Figure 2.10, we have $\lim_{z\to z_0} \operatorname{Arg} z = \pi = \operatorname{Arg} z_0$. But if z approaches z_0 from the third quadrant, say along curve C' as shown in Figure 2.10, we have $\lim_{z\to z_0} \operatorname{Arg} z = -\pi$.

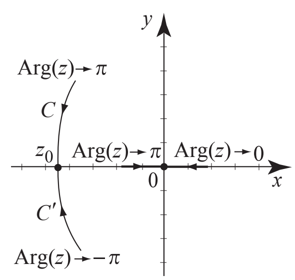

Fig. 2.10 $\operatorname{Arg} z$ has nonremovable discontinuities at $z = 0$ and at all negative real z.

Hence $\operatorname{Arg} z$ is not continuous at z_0 and the discontinuity is not removable, because $\lim_{z\to z_0} \operatorname{Arg} z$ does not exist for such z_0. It is not hard to show, using geometric considerations, that for $z \neq 0$ and z not on the negative x-axis, $\operatorname{Arg} z$ is continuous. Since the set of points of continuity of $\operatorname{Arg} z$ is the complex plane $\mathbb{C}$ minus the interval $(-\infty, 0]$ on the real line, the principal branch of the argument is continuous on $\mathbb{C} \setminus (-\infty, 0]$. $\square$

Many important functions of several variables are made up of products, quotients, and linear combinations of functions of a single variable. For example, the function $u(x, y) = e^x \cos y$ is the product of two functions of a single variable each; namely, e^x and $\cos y$. The exponential function $e^z = e^x(\cos y + i \sin y)$ is a linear combination of two products of functions of a single variable. In establishing the continuity of such functions, the following simple observations are very useful.