Solution. (a) Since we are concerned with the behavior of the function for $|z|$ large, it is safe to divide both numerator and denominator of $\frac{z-1}{z+i}$ by z, and we conclude

$$\begin{aligned}
\lim_{z\to\infty}\frac{z-1}{z+i} &= \lim_{z\to\infty}\frac{1-\frac{1}{z}}{1+\frac{i}{z}} \\
&= \frac{1-\lim\limits_{z\to\infty}\frac{1}{z}}{1+i\lim\limits_{z\to\infty}\frac{1}{z}} \qquad \text{[by (2.2.5) and (2.2.3)]} \\
&= 1 \qquad\qquad \text{[by (2.2.13)].}
\end{aligned}$$

(b) Dividing both numerator and denominator by z^2 we write

$$\lim_{z\to\infty}\frac{2z+3i}{z^2+z+1} = \lim_{z\to\infty}\frac{\frac{2}{z}+\frac{3i}{z^2}}{1+\frac{1}{z}+\frac{1}{z^2}} = \frac{0+0}{1+0+0} = 0. \qquad \square$$

While we have successfully used skills from calculus to compute complex-valued limits, real-variable intuition may not always apply. For example, the limit $\lim_{z\to\infty} e^{-z}$ is not 0; in fact, this limit does not exist (Exercise 21).

Continuous Functions

Often, the limit of a function as the variable approaches a point equals with the value of the function at this point. This property is called continuity.

Definition 2.2.12. Let f be defined on ~~an~~ a subset S of $\mathbb{C}$ and let z_0 be a point in S. We say that f is **continuous at** z_0 if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$z \in S, \quad |z-z_0| < \delta \implies |f(z)-f(z_0)| < \varepsilon. \tag{2.2.14}$$

The function f is called **continuous on** S if it is continuous at every point in S.

If z_0 is not an accumulation point of S there is a $\delta > 0$ with $B'(z_0,\delta)\cap S = \emptyset$; then $z \in S$ and $|z-z_0| < \delta \implies z = z_0$, thus f is continuous at z_0, as (2.2.14) is satisfied for any $\varepsilon > 0$. If $z_0 \in S$ happens to be an accumulation point of S, then (2.2.1) [with $L = f(z_0)$] is equivalent to (2.2.14), since obviously (2.2.14) holds when $z = z_0$; in this case f is continuous at z_0 if and only if $\lim_{z\to z_0} f(z) = f(z_0)$.

Since continuity is defined in terms of limits, many properties of limits extend to continuous functions.

Theorem 2.2.13. *Let f, g be complex-valued functions defined on a subset S of $\mathbb{C}$ and let z_0 be a point in S. Suppose that f,g are continuous at z_0. Let c_1, c_2 be complex constants. Then the following assertions are valid:*
(i) $\operatorname{Re} f$ *and* $\operatorname{Im} f$ *are continuous at* z_0.