We have avoided thus far dealing with limits that involve ∞. What do we mean by statements such as $\lim_{z\to z_0} f(z) = \infty$ or $\lim_{z\to\infty} f(z) = L$ or even $\lim_{z\to\infty} f(z) = \infty$? We answer these questions by introducing the following definitions.

Definition 2.2.10. (Limits Involving Infinity)
(i) If f is defined in a deleted neighborbood of z_0 we write $\lim_{z\to z_0} f(z) = \infty$ to mean that for every $M > 0$ there is a $\delta > 0$ such that $0 < |z - z_0| < \delta \Rightarrow |f(z)| > M$.
(ii) If f is defined in the complement of a disk centered at the origin we write $\lim_{z\to\infty} f(z) = L$ to mean that for every $\varepsilon > 0$ there is an $R > 0$ such that $|z| > R \Rightarrow |f(z) - L| < \varepsilon$.
(iii) If f is defined in the complement of a disk centered at the origin we write $\lim_{z\to\infty} f(z) = \infty$ to mean that for every $M > 0$ there is an $R > 0$ such that $|z| > R \Rightarrow |f(z)| > M$.

Looking at these definitions, we see that $z \to \infty$ means that the real quantity $|z| \to \infty$, and similarly $f(z) \to \infty$ means that $|f(z)| \to \infty$. Hence

$$\lim_{z\to z_0} f(z) = \infty \quad \Leftrightarrow \quad \lim_{z\to z_0} |f(z)| = \infty; \tag{2.2.8}$$

$$\lim_{z\to\infty} f(z) = L \quad \Leftrightarrow \quad \lim_{|z|\to\infty} |f(z) - L| = 0; \tag{2.2.9}$$

$$\lim_{z\to\infty} f(z) = \infty \quad \Leftrightarrow \quad \lim_{|z|\to\infty} |f(z)| = \infty. \tag{2.2.10}$$

Limits at infinity can also be reduced to limits at $z_0 = 0$ by means of the inversion $1/z$. The idea is that taking the limit as $z \to \infty$ of $f(z)$ is the same procedure as taking the limit as $z \to 0$ of $f\left(\frac{1}{z}\right)$. It is straightforward to verify that

$$\lim_{z\to\infty} f(z) = L \quad \Leftrightarrow \quad \lim_{z\to 0} f\left(\frac{1}{z}\right) = L; \tag{2.2.11}$$

and

$$\lim_{z\to\infty} f(z) = \infty \quad \Leftrightarrow \quad \lim_{z\to 0} f\left(\frac{1}{z}\right) = \infty. \tag{2.2.12}$$

These equivalent statements are sometimes useful. For example, appealing to (2.2.12), we have

$$\lim_{z\to\infty} \frac{1}{z} = \lim_{z\to 0} \frac{1}{1/z} = \lim_{z\to 0} z = 0. \tag{2.2.13}$$

Similarly, for a constant c and a positive integer n, we have

$$\lim_{z\to\infty} \frac{c}{z^n} = \lim_{z\to 0} \frac{c}{1/z^n} = \lim_{z\to 0} cz^n = 0.$$

Example 2.2.11. Evaluate:
(a) $\displaystyle\lim_{z\to\infty} \frac{z-1}{z+i}$; and (b) $\displaystyle\lim_{z\to\infty} \frac{2z+3i}{z^2+z+1}$.