Geometrically, interpreting the absolute value $|f(z)-L|$ as the distance between $f(z)$ and L, we see from (2.2.1) that the function $f(z)$ has limit L as $z \to z_0$ if and only if the distance from $f(z)$ to L tends to zero as z tends to z_0. See Figure 2.8. Thus, $\lim_{z\to z_0} f(z) = L$ if and only if

$$\lim_{z\to z_0} |f(z)-L| = 0. \tag{2.2.2}$$

Note that in (2.2.1) the function f need not be defined at z_0.

Proposition 2.2.2. (Uniqueness of Limits) *If f is defined on a subset S of $\mathbb{C}$, z_0 is an accumulation point of S, and $\lim_{z\to z_0} f(z)$ exists, then it is ~~a~~ unique.*

Proof. Suppose that $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} f(z) = L'$, where L, L' are different complex numbers. Then for $\varepsilon = |L-L'|/2 > 0$ there is a $\delta > 0$ such that

$$0 < |z-z_0| < \delta \implies |f(z)-L| < \frac{1}{2}|L-L'|$$
$$0 < |z-z_0| < \delta \implies |f(z)-L'| < \frac{1}{2}|L-L'|.$$

Adding, we obtain that for $0 < |z-z_0| < \delta$ we must have

$$|L-L'| \le |f(z)-L| + |f(z)-L'| < \frac{1}{2}|L-L'| + \frac{1}{2}|L-L'| = |L-L'|,$$

which is a contradiction since $|L-L'| < |L-L'|$ is impossible. ■

Example 2.2.3. Prove that:
(a) $\lim\limits_{z\to z_0} z = z_0$ (b) $\lim\limits_{z\to z_0} c = c$, where c is a constant.

Solution. (a) Given $\varepsilon > 0$, we want to find a $\delta > 0$ so that

$$0 < |z-z_0| < \delta \quad \Rightarrow \quad |f(z)-L| < \varepsilon.$$

Identifying $f(z) = z$ and $L = z_0$, this becomes

$$0 < |z-z_0| < \delta \quad \Rightarrow \quad |z-z_0| < \varepsilon.$$

Clearly, the choice $\delta = \varepsilon$ will do.
(b) The inequality $|f(z)-L| = |c-c| < \varepsilon$ holds for any choice of $\delta > 0$, and this shows that $\lim_{z\to z_0} c = c$. □

Definition 2.2.4. A function g is called **bounded** on a set S if there is a positive real number M such that $|g(z)| \le M$ for all z in S.

For example, the function $3z+2+i$ is bounded on the disk $S = \{z : |z| < 5\}$ since