Definition 2.1.6. A nonempty open and polygonally connected subset of the complex plane is called a **region**.[2]

Polygonally connected open sets are exactly the connected sets known from topology. This characterization is proved now.

Proposition 2.1.7. (Characterization of Regions) *A nonempty open subset Ω of the complex plane is polygonally connected if and only if it cannot be written as the disjoint union of two nonempty open subsets. Equivalently, if Ω is a region and $\Omega = A \cup B$, where A and B are open and disjoint, then either $A = \emptyset$ or $B = \emptyset$.*

Proof. It is best to prove the equivalence of the negations of the claimed assertions. Suppose we can write $\Omega = A \cup B$, where A, B are disjoint nonempty open sets. If $L(t)$ is a polygonal line that joins a point $z_0 \in A$ to a point $z_m \in B$, note that the set $\{t \in [0,m] : L(t) \in A\}$ is nonempty as it contains 0 and is bounded above by m. We define

$$t_* = \sup\{t \in [0,m] : L(s) \in A \text{ for all } s \in [0,t]\}.$$

Now $L(t_*)$ should lie in A or B. If $L(t_*)$ lies in B, then by the openness of B, $A \cap B \neq \emptyset$. If $L(t_*)$ lies in A, then by the openness of A, there is another point $t' > t_*$ with $L(t') \in A$, contradicting the definition of t_*. Hence we obtain that the polygonal line L is not contained in $\Omega = A \cup B$ and consequently, Ω is not polygonally connected.

Conversely, suppose that Ω is not polygonally connected. Then there exist $z_0, z_1 \in \Omega$ that are not connected via a polygonal line. We show that there exist open disjoint nonempty sets A and B such that $\Omega = A \cup B$. We define

$$A = \{z \in \Omega : z \text{ is connected to } z_0 \text{ via a polygonal line}\}$$
$$B = \{z \in \Omega : z \text{ is not connected to } z_0 \text{ via a polygonal line}\}.$$

Since $z_0 \in A$, $z_1 \in B$, these sets are nonempty. Also A and B are open, since if z is (or is not) connected to z_0 via a polygonal line, then so does an arbitrary point w that lies in an open disk contained in Ω centered at z. ∎

In topology, connected sets in the plane are exactly the ones that cannot be written as a union of two disjoint nonempty open sets. Proposition 2.1.7 indicates that open sets are connected if and only if they are polygonally connected. This characterization provides an intuitive way to determine which open subsets of the complex plane are regions.

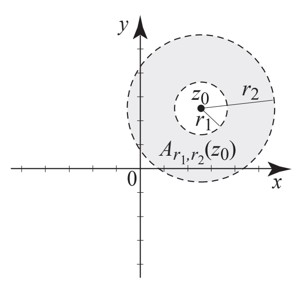

Fig. 2.6 The open annulus $A_{r_1,r_2}(z_0)$ is a region.

[2] some authors prefer the term **domain** for regions