If $1 \le p \le \infty$, Hölder's inequality gives that

$$\|\partial^\alpha u\|_{L^\infty} \le t^{|\alpha|+\frac{n}{p}} \|u\|_{L^p} \|\partial^\alpha \Phi\|_{L^{p'}} .$$

When $0 < p < 1$ we use the alternative expression for (2.2.11) which says

$$\partial^\alpha u(x) = \int_{\mathbf{R}^n} t^n \Phi(t(x-y)) \partial^\alpha u(y)\, dy$$

for all $x \in \mathbf{R}^n$. This implies that

$$|\partial^\alpha u(x)| \le t^n \|\Phi\|_{L^\infty} \|\partial^\alpha u\|_{L^\infty}^{1-p} \|\partial^\alpha u\|_{L^p}^{1-p}. \tag{2.2.12}$$

We take the supremum in (2.2.12) over $x \in \mathbf{R}^n$ and divide by $\|\partial^\alpha u\|_{L^\infty}$. We obtain

$$\|\partial^\alpha u\|_{L^\infty} \le t^{\frac{n}{p}} \|\Phi\|_{L^\infty}^{\frac{1}{p}} \|\partial^\alpha u\|_{L^p} ,$$

which yields (2.2.9), in view of (2.2.8). □

2.2.3 Equivalence of Function Space Norms

We now derive other consequences of Lemma 2.2.3 that will allow us to prove that different norms in Triebel–Lizorkin spaces are equivalent.

Corollary 2.2.5. *Let $\Phi, \Omega, \Psi \in \mathscr{S}(\mathbf{R}^n)$. Suppose that $\widehat{\Omega}(\xi), \widehat{\Psi}(\xi)$ are supported in the annulus $1 - \frac{1}{7} \le |\xi| \le 2$ and that $\widehat{\Phi}(\xi)$ is supported in the ball $|\xi| \le 2$. Let $0 < r < \infty$. Then for all f in $\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}(\mathbf{R}^n)$ and for all $x \in \mathbf{R}^n$ and $t > 0$ we have*

$$|\Phi_t * \Delta_j^\Psi(f)(x)| \le C_{\Phi,n,r} (M(|\Delta_j^\Psi(f)|^r)(x))^{\frac{1}{r}} . \tag{2.2.13}$$

In particular, for any $k, j \in \mathbf{Z}$ and $x \in \mathbf{R}^n$ we have

$$\left|\Delta_k^\Omega \Delta_j^\Psi(f)(x)\right| \le C_{\Omega,n,r} (M(|\Delta_j^\Psi(f)|^r)(x))^{\frac{1}{r}} . \tag{2.2.14}$$

Proof. Given r pick $N = \frac{n}{r} + n + 1$. Then $\Phi_t * \Delta_j^\Psi(f) = 0$ when $t > \frac{7}{3} 2^{-j}$ so (2.2.13) holds in this case. We may therefore suppose that $t^{-1} \ge \frac{3}{7} 2^j$. Then

$$\begin{aligned}
\left|(\Phi_t * \Delta_j^\Psi(f))(x)\right| &\le C_{\Phi,N} \int_{\mathbf{R}^n} \frac{\left|\Delta_j^\Psi(f)(x-z)\right|}{(1+t^{-1}|z|)^{\frac{n}{r}}} \frac{t^{-n} dz}{(1+t^{-1}|z|)^{N-\frac{n}{r}}} \\
&\le C'_{\Phi,n,r} \sup_{z \in \mathbf{R}^n} \frac{\left|\Delta_j^\Psi(f)(x-z)\right|}{(1+\frac{3}{7}2^j|z|)^{\frac{n}{r}}} \int_{\mathbf{R}^n} \frac{t^{-n} dz}{(1+t^{-1}|z|)^{N-\frac{n}{r}}} \\
&\le C_{\Phi,n,r} (M(|\Delta_j^\Psi(f)|^r)(x))^{\frac{1}{r}} ,
\end{aligned} \tag{2.2.15}$$