satisfies (2.2.1) and Θ is defined in terms of Ω in the same way that Φ is defined in terms of Ψ, [i.e., via (2.2.2)], then the norms defined in Definition 2.2.1 with respect to the pairs (Φ,Ψ) and (Θ,Ω) are comparable. To prove this assertion we need the following lemma.

Lemma 2.2.3. *Let $0<r<\infty$. Then there exist constants C_1 and C_2 such that for all $t>0$ and for all $\mathscr{C}^1$ functions u on $\mathbf{R}^n$ whose distributional Fourier transform is supported in the ball $|\xi|\le t$ we have*

$$\sup_{z\in\mathbf{R}^n}\frac{1}{t}\frac{|\nabla u(x-z)|}{(1+t|z|)^{\frac{n}{r}}}\le C_1\sup_{z\in\mathbf{R}^n}\frac{|u(x-z)|}{(1+t|z|)^{\frac{n}{r}}}\,, \tag{2.2.6}$$

$$\sup_{z\in\mathbf{R}^n}\frac{|u(x-z)|}{(1+t|z|)^{\frac{n}{r}}}\le C_2\,M(|u|^r)(x)^{\frac{1}{r}}\,, \tag{2.2.7}$$

where M denotes the Hardy–Littlewood maximal operator. The constants C_1 and C_2 depend only on the dimension n and r; in particular they are independent of t.

Proof. Select a Schwartz function Φ whose Fourier transform is supported in the ball $|\xi|\le 2$ and is equal to 1 on the unit ball $|\xi|\le 1$. Then $\widehat{\Phi}(\frac{\xi}{t})$ is equal to 1 on the support of $\widehat{u}$ and we can write

$$u(x-z)=(\Phi_{1/t}*u)(x-z)=\int_{\mathbf{R}^n}t^n\Phi(t(x-z-y))u(y)\,dy\,.$$

Taking partial derivatives and using that Φ is a Schwartz function, we obtain

$$|\nabla u(x-z)|\le C_N\int_{\mathbf{R}^n}t^{n+1}(1+t|x-z-y|)^{-N}|u(y)|\,dy\,,$$

where N is arbitrarily large. Using that for all $x,y,z\in\mathbf{R}^n$ we have

$$1\le(1+t|x-z-y|)^{\frac{n}{r}}\frac{(1+t|z|)^{\frac{n}{r}}}{(1+t|x-y|)^{\frac{n}{r}}}\,,$$

we obtain

$$\frac{1}{t}\frac{|\nabla u(x-z)|}{(1+t|z|)^{\frac{n}{r}}}\le C_N\int_{\mathbf{R}^n}t^n(1+t|x-z-y|)^{\frac{n}{r}-N}\frac{|u(y)|}{(1+t|x-y|)^{\frac{n}{r}}}\,dy\,,$$

from which (2.2.6) follows easily by choosing $N=n+1+n/r$.

We now turn to the proof of (2.2.7). We first prove this estimate under the additional assumption that u is a bounded function. Let $|y|\le\delta$ for some $\delta>0$ to be chosen later. We now apply the mean value theorem to write

$$u(x-z)=(\nabla u)(x-z-\xi_y)\cdot y+u(x-z-y)$$