$$\leq \frac{C_n B^2}{\lambda^2}\int_{\Omega_\lambda}\sum_{j=1}^{L}|g_j(x)|^2\,dx + \frac{C_n B^2}{\lambda^2}\int_{(\Omega_\lambda)^c}\sum_{j=1}^{L}|f_j(x)|^2\,dx$$

$$\leq B^2 C_{N,n}\gamma^2\,|\Omega_\lambda| + \frac{C_n B^2}{\lambda^2}\int_{(\Omega_\lambda)^c}\mathscr{M}_N(\vec{f})(x)^2\,dx\,,$$

where we used Corollary 2.1.12 in [156], the L^2 boundedness of the Hardy–Littlewood maximal operator, hypothesis (2.1.63), the fact that $f_j = g_j$ on $(\Omega_\lambda)^c$, estimate (2.1.66), and the fact that $\|\vec{f}\|_{\ell^2_L} \leq \mathscr{M}_N(\vec{f})$ in the preceding sequence of estimates.

On the other hand, estimate (2.1.71) gives

$$\big|\{M(\vec{K}*\vec{b};\Phi) > \tfrac{\lambda}{2}\}\big| \leq |\Omega_\lambda| + 2C_{N,n}A\,\gamma|\Omega_\lambda|\,,$$

which, combined with the previously obtained estimate for $\vec{g}$, gives

$$\big|\{M(\vec{K}*\vec{f};\Phi) > \lambda\}\big| \leq 2C_{N,n}(1+A\,\gamma+B^2\,\gamma^2)\,|\Omega_\lambda| + \frac{C_n B^2}{\lambda^2}\int_{(\Omega_\lambda)^c}\mathscr{M}_N(\vec{f})(x)^2\,dx\,.$$

Multiplying this estimate by $p\lambda^{p-1}$, recalling that $\Omega_\lambda = \{\mathscr{M}_N(\vec{f}) > \gamma\lambda\}$, and integrating in λ from 0 to ∞, we can easily obtain

$$\big\|M(\vec{K}*\vec{f};\Phi)\big\|^p_{L^p(\mathbf{R}^n)} \leq 2C_{N,n}(1+A\gamma+B^2\gamma^2)\gamma^{-p}\big\|\mathscr{M}_N(\vec{f})\big\|^p_{L^p(\mathbf{R}^n)}\,. \tag{2.1.72}$$

Choosing $\gamma = (A+B)^{-1}$, recalling that $N = [\frac{n}{p}]+1$, and using that $\|\mathscr{M}_N(\vec{f})\|_{L^p(\mathbf{R}^n)} \leq C'(n,p)\|\vec{f}\|_{L^p(\mathbf{R}^n,\ell^2_L)}$, gives the required conclusion.

Finally, we discuss the extension of the operator (2.1.64) to the entire $H^p(\mathbf{R}^n,\ell^2_L)$. In view of Proposition 2.1.7, $L^1(\mathbf{R}^n)\cap H^p(\mathbf{R}^n)$ is dense in $H^p(\mathbf{R}^n)$. It follows that $L^1(\mathbf{R}^n,\ell^2_L)\cap H^p(\mathbf{R}^n,\ell^2_L)$ is dense in $H^p(\mathbf{R}^n,\ell^2_L)$. Indeed, given $\vec{f} = (f_1,\dots,f_L)$ in $H^p(\mathbf{R}^n,\ell^2_L)$, find sequences $h_j^{(k)}$ in $L^1(\mathbf{R}^n)$ such that $h_j^{(k)} \to f_j$ in $H^p(\mathbf{R}^n)$ as $k\to\infty$. Set $\vec{h}^{(k)} = (h_1^{(k)},\dots,h_L^{(k)})$. Then for any $\Phi\in\mathscr{S}(\mathbf{R}^n)$ with integral one we have

$$M(\vec{f}-\vec{h}^{(k)};\Phi) \leq M(f_1-\vec{h}_1^{(k)};\Phi)+\cdots+M(f_L-\vec{h}_L^{(k)};\Phi)\,.$$

Apply the L^p quasi-norm on both sides of the preceding expression and then let $k\to\infty$ to obtain the density of $L^1(\mathbf{R}^n,\ell^2_L)\cap H^p(\mathbf{R}^n,\ell^2_L)$ in $H^p(\mathbf{R}^n,\ell^2_L)$. In view of this, the operator in (2.1.64) admits a unique bounded extension from $H^p(\mathbf{R}^n,\ell^2_L)$ to $H^p(\mathbf{R}^n)$. □

Exercises

2.1.1. Prove that if v is a bounded tempered distribution and h_1,h_2 are in $\mathscr{S}(\mathbf{R}^n)$, then

$$(h_1*h_2)*v = h_1*(h_2*v).$$