it follows that (2.1.65) is at most a constant multiple of λ, since the ball $B(c(Q_k),d_k)$ meets the complement of Ω_λ. We conclude that

$$\big\|\vec{g}\big\|_{L^\infty(\Omega_\lambda,\ell^2_L)} \le C_{N,n}\,\gamma\lambda\,. \tag{2.1.66}$$

We now estimate $M(\sum_{j=1}^L K_j * b_{j,k};\Phi)$. For fixed k and $\varepsilon>0$ we have

$$\begin{aligned}
&\Big(\Phi_\varepsilon * \sum_{j=1}^L K_j * b_{j,k}\Big)(x)\\
&\quad= \int_{\mathbf{R}^n}\Big(\Phi_\varepsilon * \sum_{j=1}^L K_j\Big)(x-y)\bigg[f_j(y)\varphi_k(y) - \frac{\int_{\mathbf{R}^n} f_j\varphi_k\,dx}{I_k}\,\varphi_k(y)\bigg]dy\\
&\quad= \int_{\mathbf{R}^n}\sum_{j=1}^L\bigg\{\big(\Phi_\varepsilon * K_j\big)(x-z) - \int_{\mathbf{R}^n}\big(\Phi_\varepsilon * K_j\big)(x-y)\frac{\varphi_k(y)}{I_k}\,dy\bigg\}\varphi_k(z)f_j(z)\,dz\\
&\quad= \int_{\mathbf{R}^n}\sum_{j=1}^L R_{j,k}(x,z)\varphi_k(z)f_j(z)\,dz\,,
\end{aligned}$$

where we set $R^\varepsilon_{j,k}(x,z)$ for the expression inside the curly brackets. Using (2.1.52), we obtain

$$\begin{aligned}
&\bigg|\int_{\mathbf{R}^n}\sum_{j=1}^L R^\varepsilon_{j,k}(x,z)\varphi_k(z)f_j(z)\,dz\bigg|\\
&\qquad\le \sum_{j=1}^L \mathfrak{N}_N\big(R^\varepsilon_{j,k}(x,\cdot\,)\varphi_k;c(Q_k),d_k\big)\inf_{|z-c(Q_k)|\le d_k}\mathscr{M}_N(f_j)(z)\\
&\qquad\le \sum_{j=1}^L \mathfrak{N}_N\big(R^\varepsilon_{j,k}(x,\cdot\,)\varphi_k;c(Q_k),d_k\big)\inf_{|z-c(Q_k)|\le d_k}\mathscr{M}_N(\vec{f}\,)(z)\,.
\end{aligned} \tag{2.1.67}$$

Since $\varphi_k(z)$ is supported in $\frac{6}{5}Q_k$, the term $(1+\frac{|z-c(Q_k)|}{d_k})^N$ contributes only a constant factor in the integral defining $\mathfrak{N}_N\big(R^\varepsilon_{j,k}(x,\cdot\,)\varphi_k;c(Q_k),d_k\big)$, and we obtain

$$\begin{aligned}
&\mathfrak{N}_N\big(R^\varepsilon_{j,k}(x,\cdot\,)\varphi_k;c(Q_k),d_k\big)\\
&\qquad\le C_{N,n}\int_{\frac{6}{5}Q_k}\sum_{|\alpha|\le N+1} d_k^{|\alpha|}\Big|\frac{\partial^\alpha}{\partial z^\alpha}\big(R^\varepsilon_{j,k}(x,z)\varphi_k(z)\big)\Big|\,dz\,.
\end{aligned} \tag{2.1.68}$$

For notational convenience we set $K^\varepsilon_j = \Phi_\varepsilon * K_j$. We observe that the family $\{K^\varepsilon_j\}_j$ satisfies (2.1.62) and (2.1.63) with constants A' and B' that ~~are only~~ multiples of $A+B$ ~~respectively,~~ uniformly in ε; see Exercise 2.1.13. We now obtain a pointwise estimate for $\mathfrak{N}_N\big(R^\varepsilon_{j,k}(x,\cdot\,)\varphi_k;c(Q_k),d_k\big)$ when $x\in\mathbf{R}^n\setminus\Omega_\lambda$. For fixed $x\in\mathbf{R}^n\setminus\Omega_\lambda$ we have

$$R^\varepsilon_{j,k}(x,z)\varphi_k(z) = \int_{\mathbf{R}^n}\varphi_k(z)\Big\{K^\varepsilon_j(x-z) - K^\varepsilon_j(x-y)\Big\}\frac{\varphi_k(y)\,dy}{I_k}\,,$$